NUS – School of Computing
CS6234 Advanced Topic in Algorithms

Parallel and Distributed Algorithms

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Outline

- Background (Abdelrahman)
  - Background (1) Parallel and Distributed Algorithms
  - Background (2) Perfect Matchings
- Perfect Matchings (Abdelhak & Yifan)
- Byzantine Agreement (Dileepa & Xin)

Background (1) Parallel and Distributed Algorithms

The use and organization of multiple processors to solve a problem

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- Parallel
    - Processor share clock and memory
    - Same OS
    - Frequent communication

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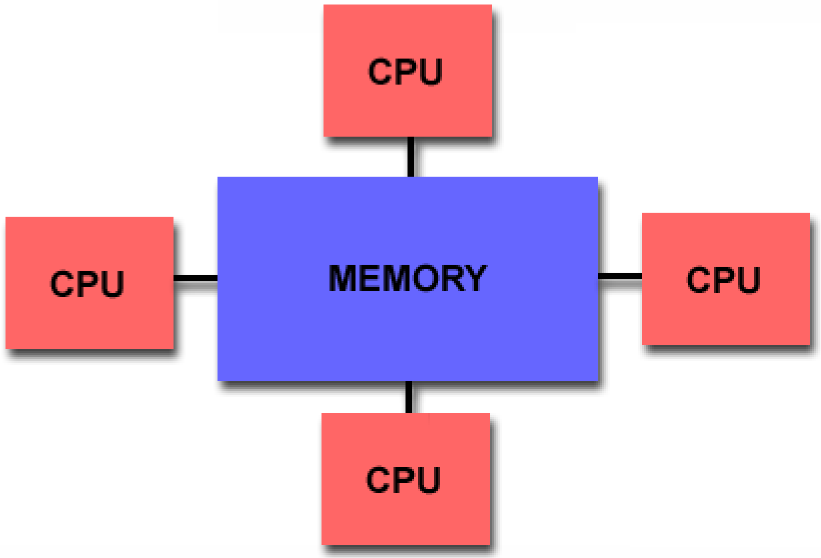

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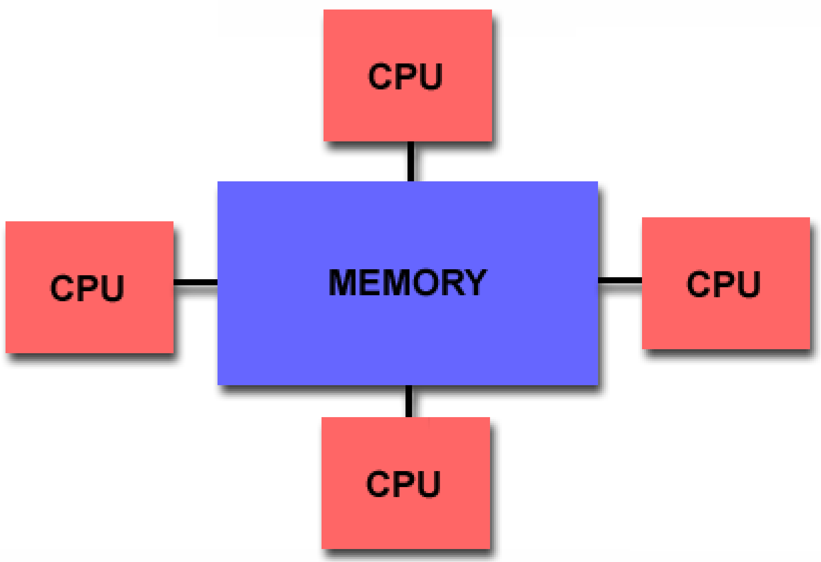

- Distributed
  - Memory not shared
  - Different clocks
  - Different OS
  - Infrequent communication

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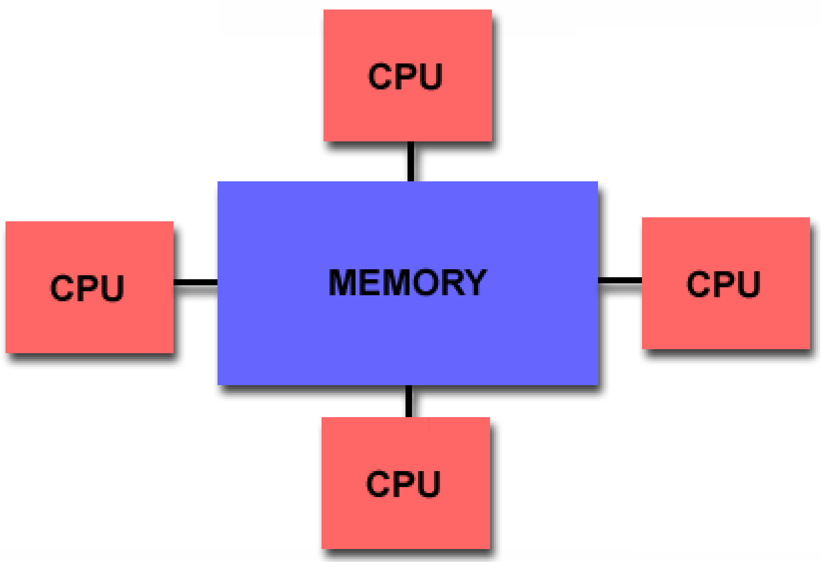

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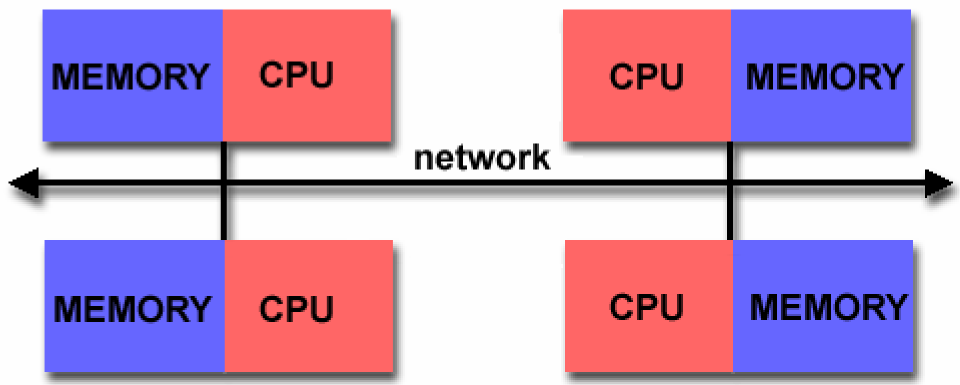
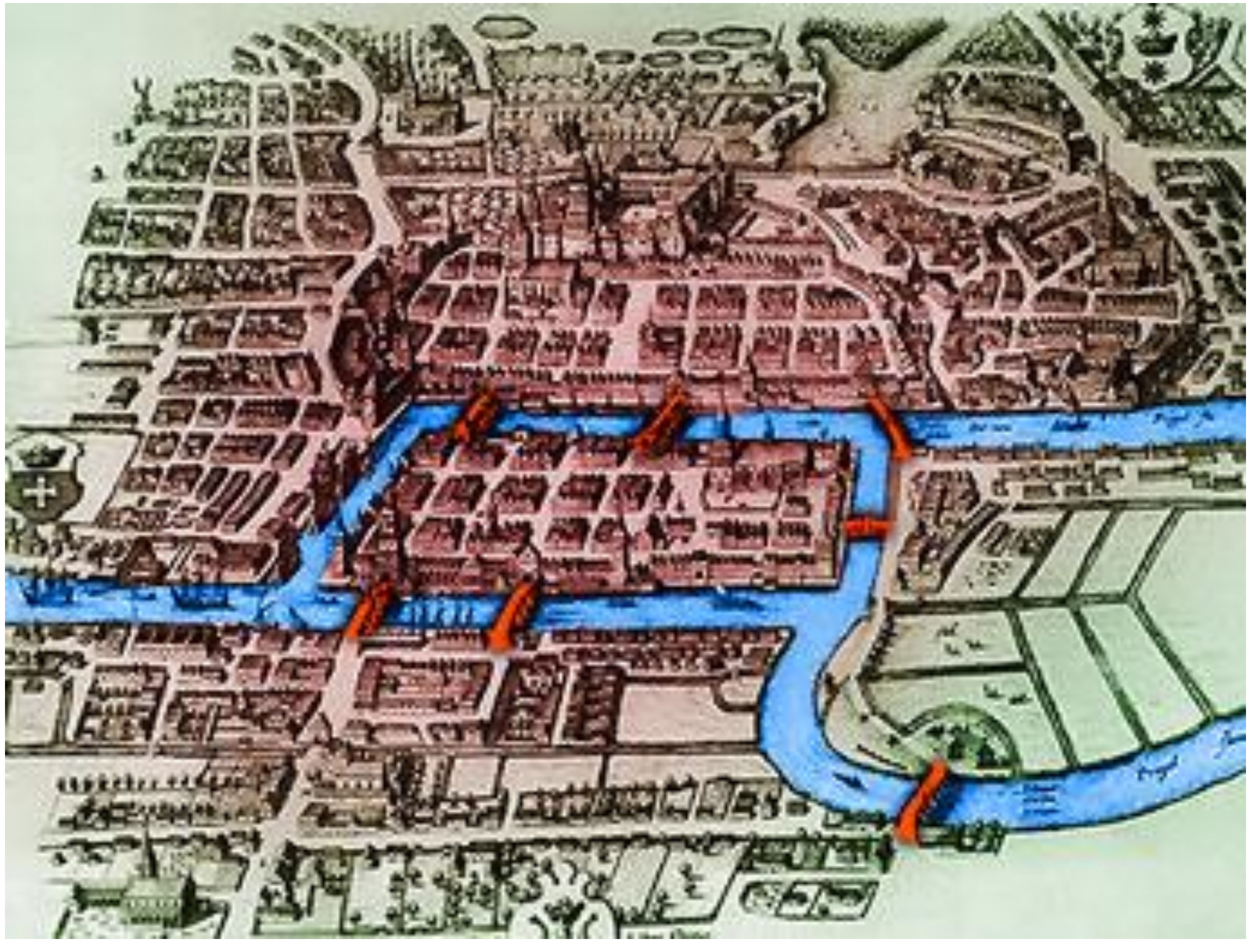


Background (2) Perfect Matchings

- The Seven Bridges of Königsberg
  - How to cross all 7 bridges exactly once?

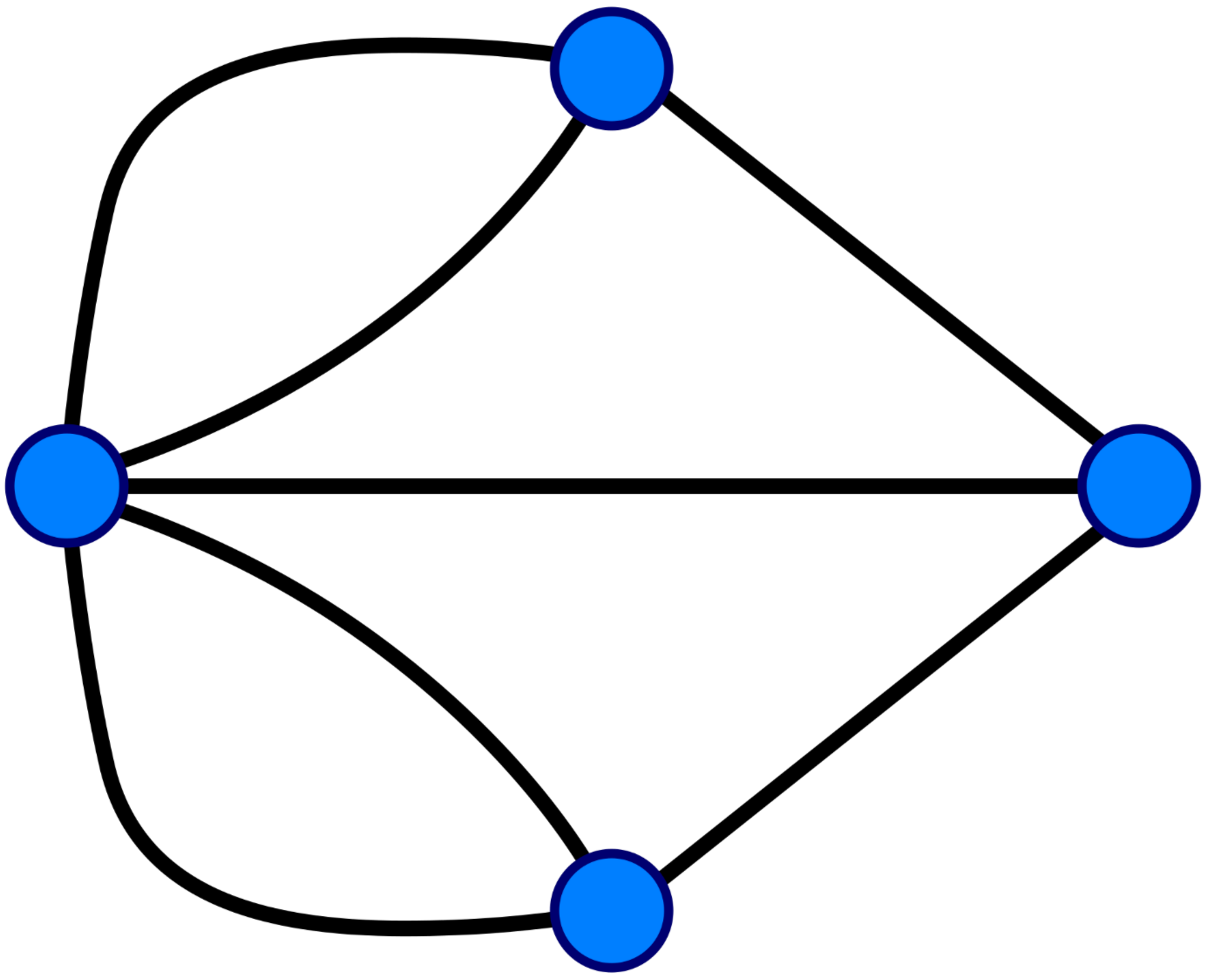


Background (2) Perfect Matchings

- The Seven Bridges of Königsberg
  - Represent regions by points
  - Represent bridges by lines

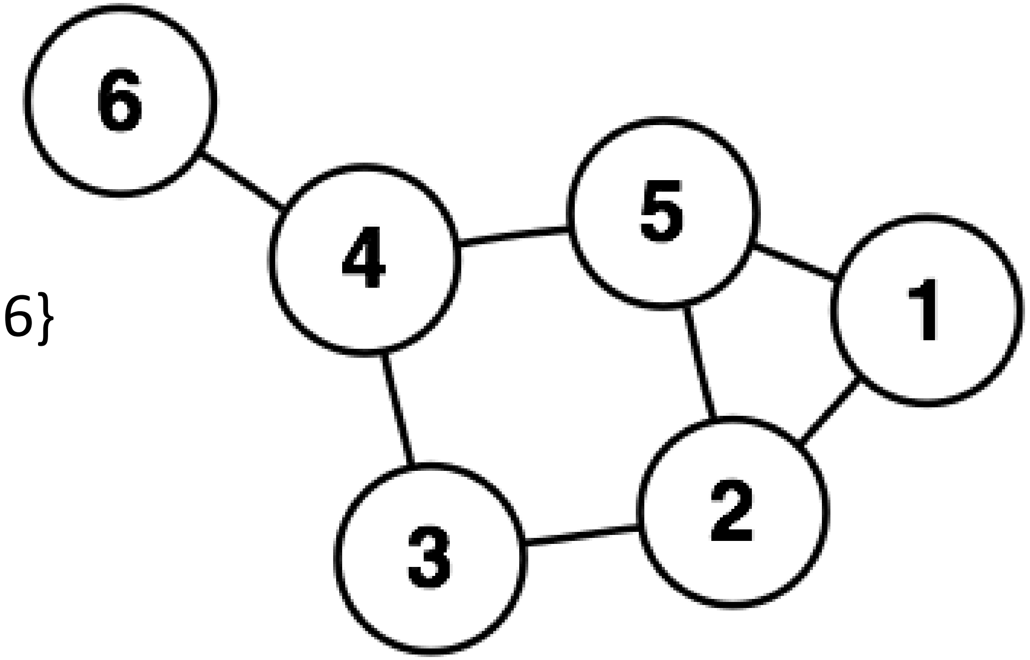
Background (2) Perfect Matchings

- The Seven Bridges of Königsberg
  - Points: Vertices
  - Lines: Edges
- This is a GRAPH



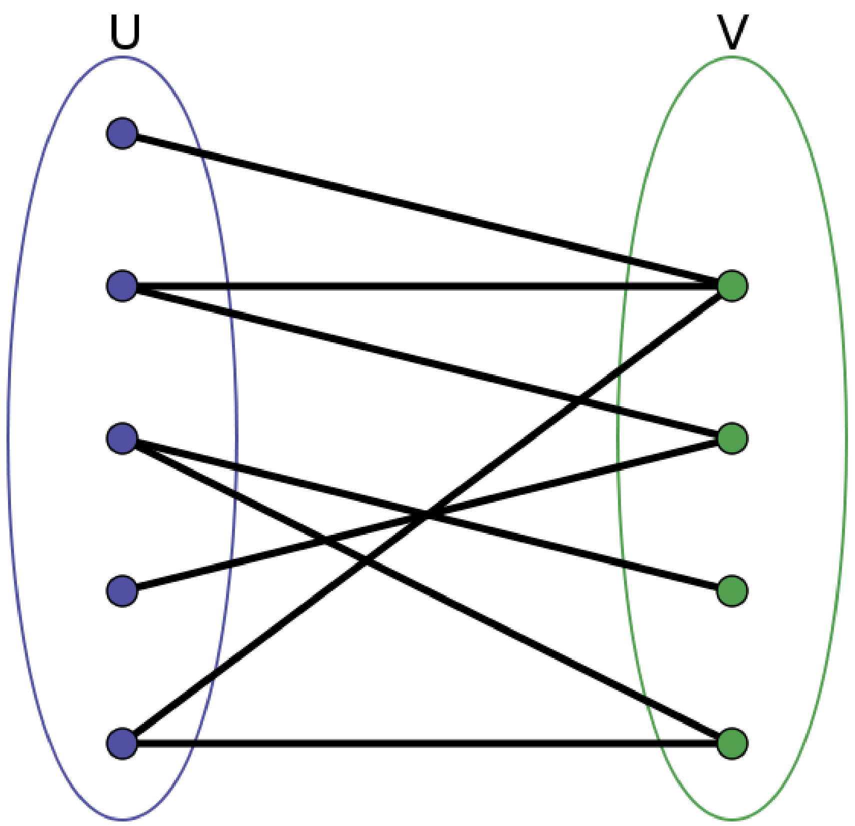
Background (2) Perfect Matchings

- Example Graph
  - 6 Vertices:
    {1, 2, 3, 4, 5, 6}
  - 7 Edges:
    {1, 2}, {1, 5}, {2, 3}, {2, 5}, {3, 4}, {4, 5}, {4, 6}



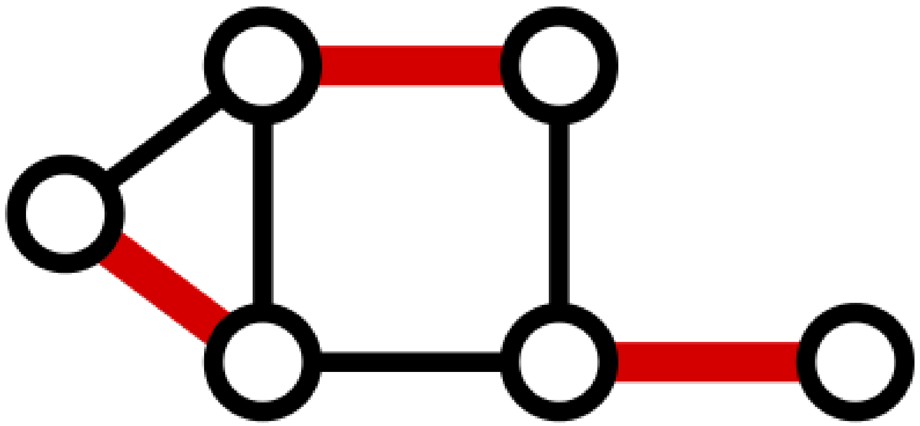
Background (2) Perfect Matchings

- Bipartite graph (bigraph)
  - a graph whose vertices can be divided into two disjoint sets U and V
  - such that every edge connects a vertex in U to one in V.



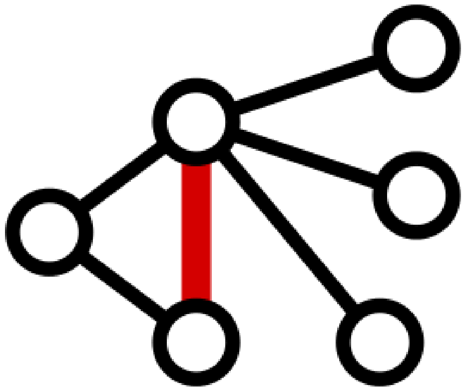
Background (2) Perfect Matchings

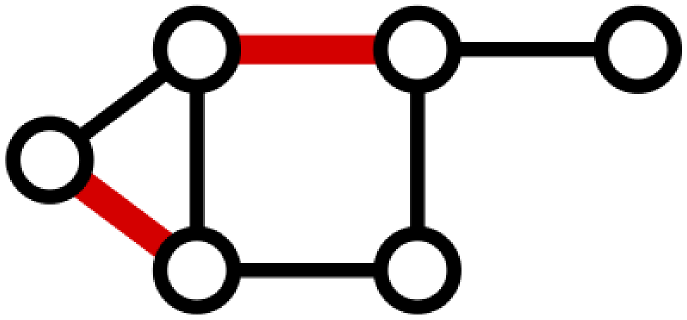
- Matching (independent edge set)
  - a set of edges M without common vertices

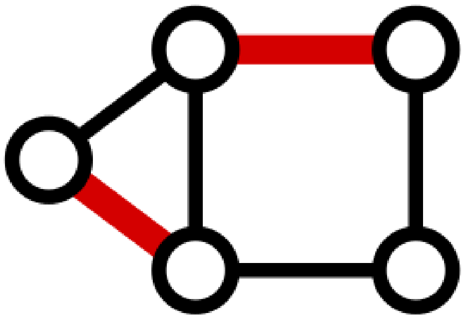


Background (2) Perfect Matchings

- Maximal matching
  - M is maximal if it is not a proper subset of any other matching in graph G
  - every edge in *G* has a non-empty intersection with at least one edge in *M*

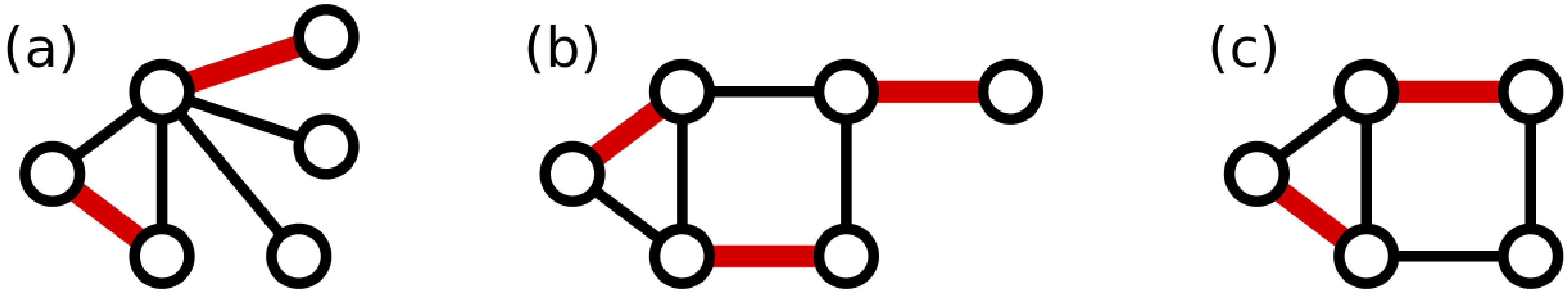






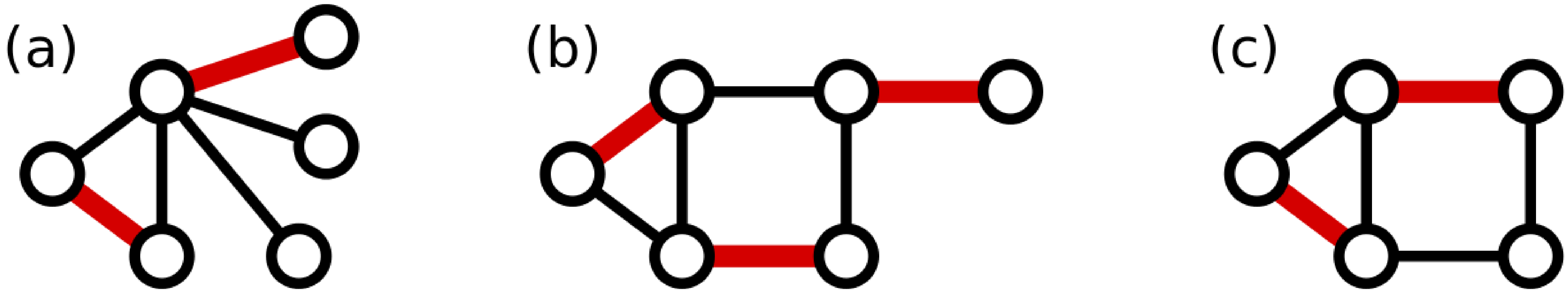
Background (2) Perfect Matchings

- Maximum matching
  - a matching that contains the largest possible number of edges



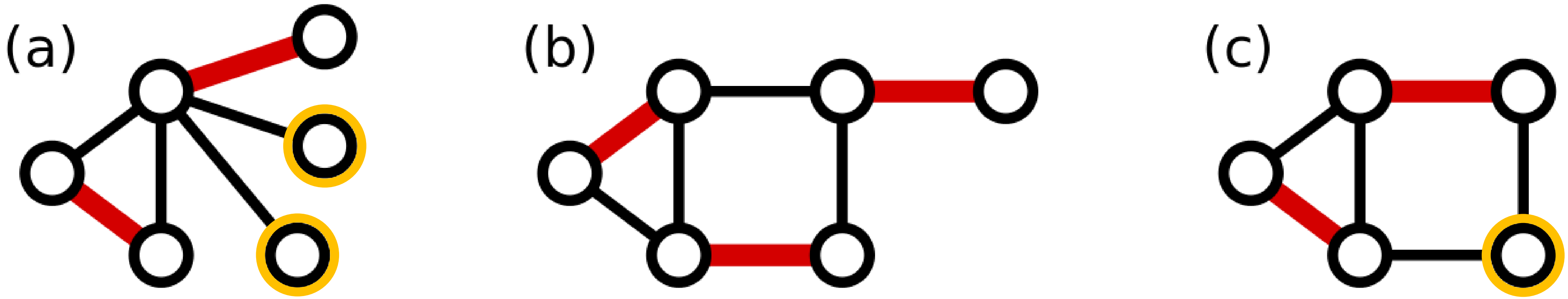
Background (2) Perfect Matchings

- Perfect matching
  - a matching that contains all vertices of the graph
  - Which of a, b, or c is a prefect matching?



Background (2) Perfect Matchings

- Perfect matching
  - a matching that contains **all vertices** of the graph
  - Which of a, b, or c is a prefect matching?



Perfect Matching

Abdelhak Bentaleb[1]

[1]School of Computing

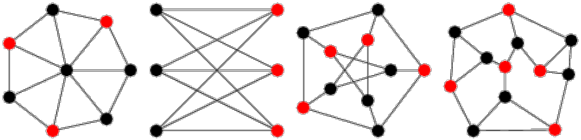
2016

Outline

Outline

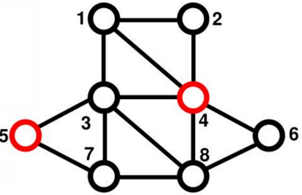
Independent Set

- Given a graph $G = (V, E)$, find a subset $I \subseteq V$ such that for all $(u, v) \in E$: $u \notin I$ or $v \notin I$.
- An independent set I is maximal if I can not be augmented to a larger independent set.
- An independent set I is **maximum** if for all independent sets I', we have that $|I| \geq |I'|$



How do we find a MIS ?

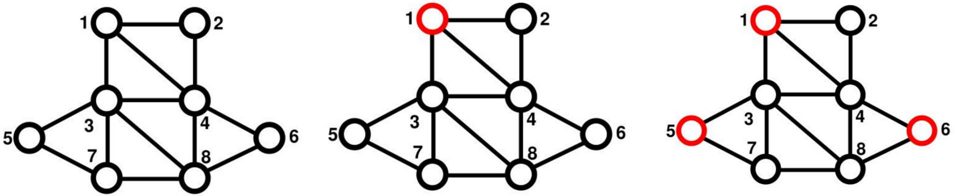
- Finding a maximum Independent Set is NP-hard.
- Finding a maximal Independent Set is simple.
- Graph with vertices V = { 1,2,,n }
- A set S of vertices is independent if no two vertices in S are neighbors.
- An independent set S is maximal if it is impossible to add another vertex and stay independent
- An independent set S is maximum if no other independent set has more vertices.



The set of red vertices S = { 4, 5} is independent and maximal but not maximum

Sequential Algorithm MIS

1. S = empty set
2. for vertex v = 1 to N
3. if (v has no neighbor in S)
4. add v to S



S = { }

S = { 1 }

S = { 1, 5, 6 }

work **~ O(n), but** ***span*** **~O(n)**

Parallel, Randomized MIS Algorithm

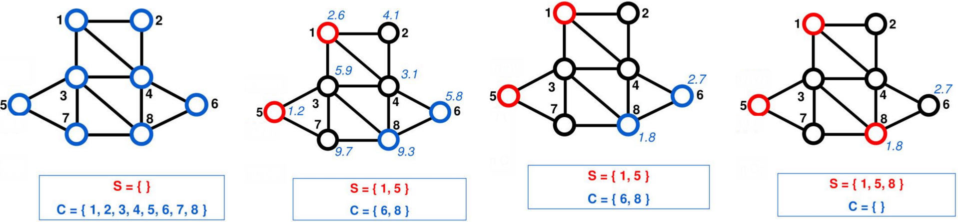
Luby

- $S \leftarrow \emptyset$, G the graph
- While G not empty do IN PARALLEL
    - Mark each vertex v independently with probability $\frac{1}{2d(v)}$ (always mark isolated nodes)
    - For every edge with both nodes marked, unmark the node with the lowest degree (break ties arbitrarily)
    - Let C be the set of all marked nodes, $S \leftarrow S \cup C$
    - Remove from G the vertices $C \cup N(C)$ and all incident edges

Parallel, Randomized MIS Algorithm

Luby

1. S = empty set, C = V
2. while C is not empty
3. label each v in C with a probability r(v) $= \dfrac{1}{2d(v)}$
4. for all v in C in parallel
5. if r(v) < min(r($\Gamma(v)$))
6. move v from C to S
7. remove neighbors of v from C

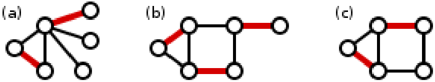


work ~ O(n log n), but *span* ~O(log n)

Outline

Matchings

- Let $G = (V, E)$ be a graph
- A matching in G is a set of edges $M \subset E$ such that no two edges are incident
- A maximum matching is a matching with maximum number of edges [c]
- A perfect matching is a matching containing an edge incident to every vertex of G [b]



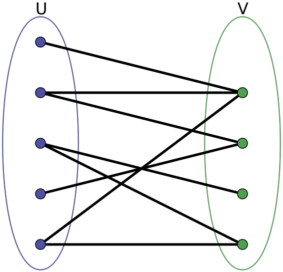
Determining the existence of a perfect matching

Bipartite Graphs

- For simplicity, we will deal with bipartite graphs such that G = (U, V, E) and U = $\{u_1 , . . . u_n \}$, V = $\{v_1 , ... , v_n\}$

Definition

A simple graph G = (V,E) is called bipartite if its vertex set can be partitioned into two disjoint V = $u_1 \cup v_1$, such that every edge has the form e = (a,b) where a $\in u_1$ and b $\in v_1$. Note, that no vertices both in u_1 or both in v_1 are connected

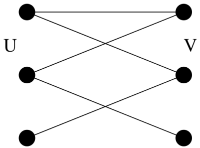


Determining the existence of a perfect matching

The Tutte Matrix

- The Tutte Matrix A of a bipartite graph G is a n $\times$ n matrix such. that:

$$A_{ij} = \begin{cases} xij & \text{if } (u_i, v_j) \in E \textit{ and } i < j \\ -xij & \text{if } (u_i, v_j) \in E \textit{ and } i > j \\ 0 & \text{if } (u_i, v_j) \notin E \end{cases}$$



$$A = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 0 & x_{23} \\ 0 & x_{32} & 0 \end{pmatrix}$$

Determining the existence of a perfect matching

Determinant of the Tutte Matrix

Theorem

$det(A) \neq 0 \Leftrightarrow$ G has a perfect matching

- det(A) is a polynomial with n^2 variables.
- Use the Schwartz-Zippel algorithm for Polynomial Identity Testing to check whether $\det(A) = 0$.
- Computing the determinant is used as a subroutine.
- A $n \times n$ determinant can be computed in $O(\log^2 n)$ time using polynomially many processors

Determining the existence of a perfect matching

Decision version of Perfect Matching

Consequence

Deciding whether a graph G has a perfect matching is in RNC (Random Nick's Class (NC)).

Note: the class NC (for "Nick's Class") is the set of decision problems decidable in polylogarithmic time on a parallel computer with a polynomial number of processors. In other words, a problem is in NC if there exist constants c and k such that it can be solved in time $O(\log^c n)$ using $O(n^k)$ parallel processors

Finding a Perfect Matching Sequentially

- Notice that if edge e belongs to a perfect matching, then for the graph G' $=$ G $\setminus$ e we have that det(A') $\neq$ 0.
- Sequential Matching
  1. Pick an arbitrary edge (i, j) of G
  2. Check whether G' $=$ G $\setminus$ i, j has a perfect matching
  3. IF YES, add edge (i, j) to the matching M and G $\leftarrow$ G'
  4. ELSE G $\leftarrow$ G $\setminus$ $\{$(i, j)$\}$.
  5. While M is not a perfect matching, repeat 1

Finding a Perfect Matching: The Parallel Algorithm

Ideas

- Not parallelizable: G may have many perfect matchings, the processors must be coordinated to search for the same matching!
- IDEA: isolate a perfect matching and then employ the algorithm
- HOW? assign random weights and look for the minimum weight matching

Isolating Lemma

Will be completed by Yifan

Lemma & parallel algorithm for perfect matching

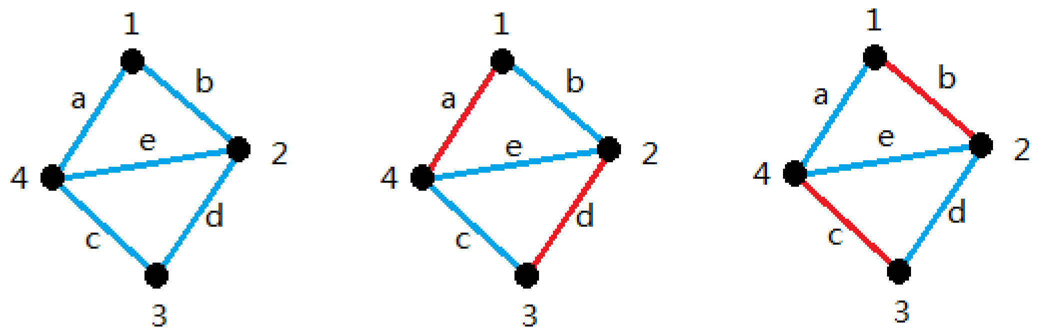
LEI YIFAN

General Idea

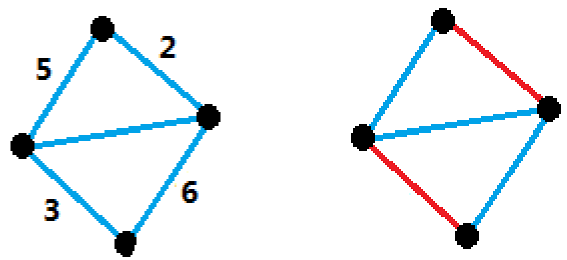
- The general idea to parallelize this sequential algorithm is let all thread find the unique minimum weight perfect matching.

Example

For the graph below, we have two different perfect matching {a,d} and {b,c}.



After assigning each edge a weight, there might be one unique minimum weight perfect matching.



Problems

- How to make sure that there is exactly one minimum perfect matching.

- How to compute the unique minimum perfect matching.

Problems

- To address these problems, we have some lemmas. Since their proof is quite complex and takes too much time, we may mainly focus on the algorithm and how the algorithm can get the right solution.

Isolating lemma

- A *set system* (X, $\mathcal{F}$) consists of a finite universe $X = \{x_1, \dots, x_m\}$ and a family of subsets $\mathcal{F} = \{S_1, \dots, S_k\}$, where $S_i \subseteq X$ for $1 \le i \le k$. The *dimension* of the set system is m.

- Lemma: Suppose (X, $\mathcal{F}$) is a set system of dimension m. Let $w: X \to \{1, \dots, 2m\}$ be a positive integer weight function defined by assigning to each element of X a random weight chosen uniformly and independently from $\{1, \dots, 2m\}$. Then,

$$\Pr[\textit{there is a unique minimum weight set in } \mathcal{F}] \ge \frac{1}{2}.$$

Example

- $X = \{1,2,3,4,5,6\}$ and m=6
- $\mathcal{F} = \{\{1,2,3\}, \{2,3,5\}, \{3,4,6\}, \{2,4,5,6\}\}$
- Randomly set $w: X \rightarrow \{1, \dots 2m\}$ as table below:

x	1	2	3	4	5	6	mininum F
1st random assignment	11	5	1	7	5	12	{2,3,5}
2nd	2	11	10	8	3	6	{1,2,3}
3rd	2	5	12	2	10	4	{3,4,6}
4th	2	10	3	3	9	9	{1,2,3}&{3, 4,6}
5th	3	10	8	11	10	10	{1,2,3}
6th	10	9	1	2	4	6	{3,4,6}

Simple Idea about Proof

- For element x_i ,let W_i be the weight of a minimum weight set containing x_i and $\overline{W_i}$ be the weight of a minimum weight set which do not contain x_i.
- Only when $w(x_i) = \overline{W_i} - W_i$, there will not be an unique minimum weight set.
- The probability of $w(x_i) = \overline{W_i} - W_i$ is no more than 1/2m.
- Since we have m element. The probability that there is a unique minimum weight set should at most

$$m \times \frac{1}{2m} = \frac{1}{2}.$$

- According to Isolating Lemma, we can assign a random weight to each edge and, with high probability, that graph would have unique minimum weight perfect matching.
- If there is no unique minimum weight perfect matching, the algorithm will fail. We can repeat until the graph do have a unique minimum weight perfect matching.

Tutte Matrix

- Let B be the matrix obtained from A by setting each indeterminate x_{ij} to the (random) integer value $2^{w_{ij}}$

- Example:

If Tutte Matrix $A = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 0 & x_{23} \\ 0 & x_{32} & 0 \end{pmatrix}$

Then $B = \begin{pmatrix} 2^{w_{11}} & 2^{w_{12}} & 0 \\ 2^{w_{21}} & 0 & 2^{w_{23}} \\ 0 & 2^{w_{32}} & 0 \end{pmatrix}$

Lemma

- There is a unique minimum weight perfect matching and that its weight is W $\Leftrightarrow$ The highest power of 2 that divides det(B) is 2^{2W}

- Proof:
- Similar to previous proof.

Lemma

- Let M be the unique minimum weight perfect matching in G, and let its weight be W. An edge (i,j) belongs to M if and only if

$$\frac{\det\left(B^{ij}\right) 2^{w_{ij}}}{2^{2W}}$$

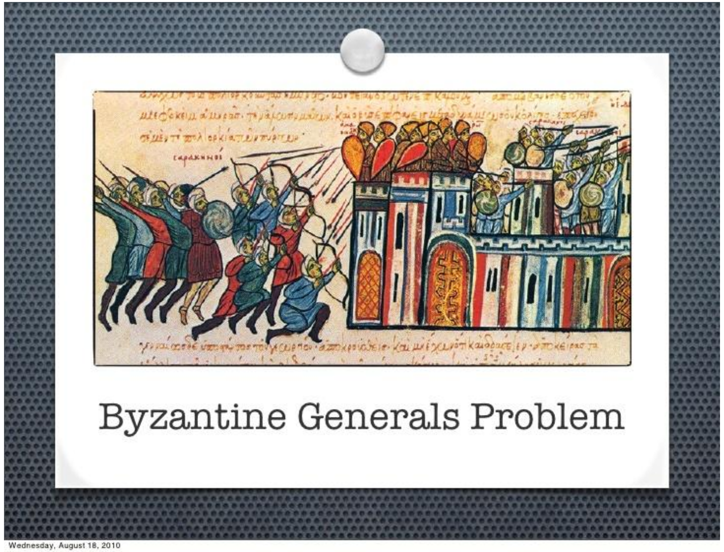
- is odd.
- Proof:
  - $\det\left(B^{ij}\right)$ is related the perfect matching of G/(i,j).

Parallel Perfect Matching

- 1. **for all** edges (i,j), **in parallel** do
  Choose random weight w_{ij}.
- 2. compute the Tutte matrix B from w.
- 3. compute det(B).
- 4. compute W such that 2^{2W} is the largest power of 2 dividing det(B).
- 5. compute $adj(B) = \det(B) \times B^{-1}$ whose (j,i) entry has absolute value det(B^{ij}).
- 6. **for all** edges (i,j) **in parallel** do
  Compute $r_{ij} = \det\left(B^{ij}\right) 2^{w_{ij}} / 2^{2W}$.
- 7. **for all** edges (i,j) **in parallel** do
  If r_{ij} is odd then add (i,j) to M

Byzantine Generals

- “The Byzantine Generals Problem”, by Lamport, Shostak, Pease, In ACM Transactions on Programming Languages and Systems, July 1982
    - why we need agreement
    - problem definition
    - impossible case
    - algorithm steps through examples



Motivation

- Build reliable systems in presence of faulty components
- Failed components send conflicting information to different parts of system
- Agreement in the presence of faults
- P2P Networks?
  - Good nodes have to “agree to do the same thing”.
  - Faulty nodes generate corrupted and misleading messages.
  - Non-malicious: Software bugs, hardware failures, power failures
  - Malicious reasons: Machine compromised.

General Problem definition

- The abstract of problem:
  - Each division of Byzantine army is directed by its own general.
  - There are n Generals, some of which are traitors.
  - All armies are camped outside enemy castle, observing enemy.
  - Communicate with each other by messengers.
- Requirements:
  - R1: All loyal generals decide upon the same plan of action
  - R2: A small number of traitors cannot cause the loyal generals to adopt a bad plan
- Note: We **do not** have to identify the traitors.

Common Approach

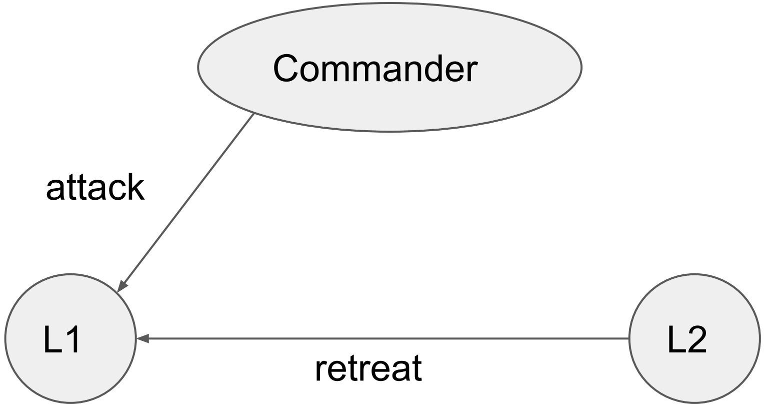
- i^{th} general sends information v(i) to all other generals
- To deal with two requirements:
  - All generals combine their information v(1), v(2), .., v(n) in the same way
  - Majority (v(1), v(2), …, v(n)), ignore minority traitors
- Common approach may not work:
  - Traitors may send different values to different generals
  - Loyal generals might get conflicting values from traitors

Reduction of General Problem

- **Insight:** We can restrict ourselves to the problem of one general sending its order to others.
- **Byzantine Generals Problem (BGP):**
  - A commanding general (commander) must send an order to his n-1 lieutenants.
- **Interactive Consistency Conditions:**
  - IC1: All loyal lieutenants obey the same order.
  - IC2: If the commanding general is loyal, then every loyal lieutenant obeys the order he sends.
- **Note**: If General is loyal, IC2 => IC1.

Impossible Case

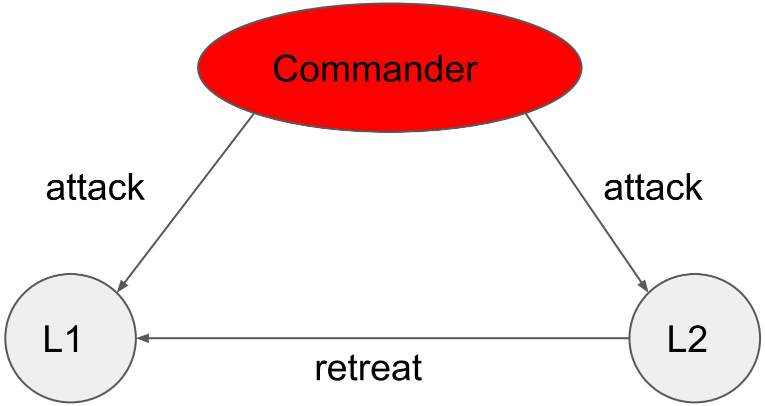
- 3 generals, 1 traitor among them.
- Two messages: Attack or Retreat



What should L1 do? Is commander or L2 the traitor???

Option 1: Loyal Commander

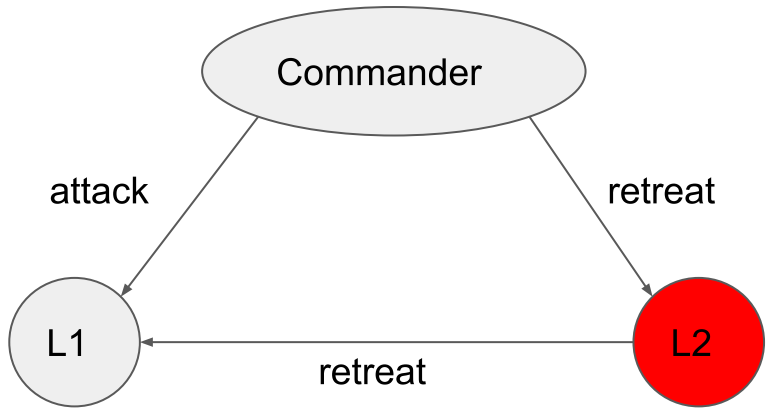
What must L1 do?



By IC2: L1 must obey commander and attack

Option 2: Loyal L2

What must L1 do?



By IC1: L1 and L2 must obey same order --> L1 must retreat

Problem: L1 can't distinguish between 2 scenarios

General Impossible Result

- No solution with fewer than 3m+1 generals can cope with m traitors
- Refer to the paper for details

Byzantine Agreement

Dileepa Fernando

Scenario

- Presence of good people and bad people (remember the mafia game?)

- All the good people should agree on the same decision at the end

- Bad people try to manipulate the decision of good people (How?)

How people behave

- Any two people have a two way communication link
- Every person knows the identities of the others
- Only bad people can distinguish bad people from good people
- Good people cannot identify the identities of bad people (initially)

Byzantine agreement

- All the people have their corresponding bit value(initial value)
- Final decision of all the good people should be the same value
- If initial value of all the good people is same, final decision should be the initial value

How do bad people violate the protocol

- Good people broadcast same value to all other people in each round

- Bad people can send different values to different people in each round and manipulate the majority value

- It is assumed that number of bad people (t) is less than n/8 (n is the total number of people)

Terms

- L= (5n)/8 + 1 (lower threshold)
- H= (3n)/4 + 1 (higher threshold)
- G= (7n)/8

- b_i Initial value of player i
- d_i decision of player i

- tally= number of occurrences of the majority vote
- maj= value of the majority vote (0 or 1)

- L ≥ n/2 + t + 1 (why?)
- H ≥ L + t (why?)

Algorithm

Algorithm ByzGen:

Input: A value b_i.

Output: A decision d_i.

1. $vote = b_i$;
2. For each round, **do**

   3. Broadcast *vote*;
   4. Receive votes from all other processors;
   5. $maj \leftarrow$ majority value (0 or 1) among votes received including own vote;
   6. $tally \leftarrow$ number of occurrences of *maj* among votes received;
   7. **if** $coin =$ HEADS **then** $threshold \leftarrow L$;
      else $threshold \leftarrow H$;
   8. **if** $tally \geq threshold$ **then** $vote \leftarrow maj$;
      else $vote \leftarrow 0$;
   9: **if** $tally \geq G$ **then** set d_i to *maj* permanently;

Example I

- Suppose 8 players (L=6, H=7, G=7)
- Only one bad player.
- What if all the good players broadcast ‘1’?

Example set of row vectors after 1st round

	1	2	3	4	5	6	7	8	Tally
1	1	1	1	1	1	1	1	1	8
2	1	1	1	1	1	1	1	0	7
3	1	1	1	1	1	1	1	1	8
4	1	1	1	1	1	1	1	0	7
5	1	1	1	1	1	1	1	1	8
6	1	1	1	1	1	1	1	0	7
7	1	1	1	1	1	1	1	1	8
8	1	1	1	1	1	1	1	0	7

8 is the bad player, for all good players, tally $\geq G$

Algorithm

Algorithm ByzGen:

Input: A value b_i.

Output: A decision d_i.

1. $vote = b_i$;
2. For each round, **do**

    3. Broadcast *vote*;
    4. Receive votes from all other processors;
    5. $maj \leftarrow$ majority value (0 or 1) among votes received including own vote;
    6. $tally \leftarrow$ number of occurrences of *maj* among votes received;
    7. **if** $coin = \text{HEADS}$ **then** $threshold \leftarrow L$;
       else $threshold \leftarrow H$;
    8. **if** $tally \geq threshold$ **then** $vote \leftarrow maj$;
       else $vote \leftarrow 0$;
    9: **if** $tally \geq G$ **then** set d_i to *maj* permanently;

Example II

- What if different players have different tally values?

- Intuition – only bad players can make this change, the difference is bounded, nearly balanced vote can be changed

- If n_1 is the tally value, $n/2 \le n_1 \le n/2 + t$ (Can you prove this?)

- What if lowest threshold is greater than the greatest tally value possible (Remember $L \ge n/2 + t + 1$)

Example set of row vectors after 1^{st} round

	1	2	3	4	5	6	7	8	Tally
1	1	1	1	1	0	0	0	1	5
2	1	1	1	1	0	0	0	0	4
3	1	1	1	1	0	0	0	1	5
4	1	1	1	1	0	0	0	0	4
5	1	1	1	1	0	0	0	1	5
6	1	1	1	1	0	0	0	0	4
7	1	1	1	1	0	0	0	1	5
8	1	1	1	1	0	0	0	0	4

8 is the bad player, for all good players, tally $\leq$ threshold

Algorithm

Algorithm ByzGen:

Input: A value b_i.

Output: A decision d_i.

1. $vote = b_i$;
2. For each round, **do**
   3. Broadcast *vote*;
   4. Receive votes from all other processors;
   5. $maj \leftarrow$ majority value (0 or 1) among votes received including own vote;
   6. $tally \leftarrow$ number of occurrences of *maj* among votes received;
   7. **if** $coin =$ HEADS **then** $threshold \leftarrow L$;
      else $threshold \leftarrow H$;
   8. **if** $tally \geq threshold$ **then** $vote \leftarrow maj$;
      else $vote \leftarrow 0$;
   9: **if** $tally \geq G$ **then** set d_i to *maj* permanently;

Example III

- Two cases were covered in previous example.

What is the remaining case?

All players have vectors with same majority value.

Can there be two tally values for two vectors, $t1 \leq L$ and $t2 > H$ in this case? (Remember $H \geq L + t$)

Example 3 (contd)

- all tally values ≤ L,H (same as Example I)

or

- all tally values ≤ H

or

- all tally values > L

Tally values has a lower bound or upper bound

What if this bound(L or H) is generated from the coin toss? (what is the probability?)

Example set of row vectors after 1^{st} round

	1	2	3	4	5	6	7	8	tally
1	1	1	1	1	1	1	0	1	7
2	1	1	1	1	1	1	0	0	6
3	1	1	1	1	1	1	0	1	7
4	1	1	1	1	1	1	0	0	6
5	1	1	1	1	1	1	0	1	7
6	1	1	1	1	1	1	0	0	6
7	1	1	1	1	1	1	0	1	7
8	1	1	1	1	1	1	0	0	6

8 is the bad player, for all good players, tally $\geq L$

Algorithm

Algorithm ByzGen:

Input: A value b_i.

Output: A decision d_i.

1. $vote = b_i$;
2. For each round, **do**
    3. Broadcast *vote*;
    4. Receive votes from all other processors;
    5. $maj \leftarrow$ majority value (0 or 1) among votes received including own vote;
    6. $tally \leftarrow$ number of occurrences of *maj* among votes received;
    7. **if** $coin =$ HEADS **then** $threshold \leftarrow L$;
       else $threshold \leftarrow H$;
    8. **if** $tally \geq threshold$ **then** $vote \leftarrow maj$;
       else $vote \leftarrow 0$;
    9: **if** $tally \geq G$ **then** set d_i to *maj* permanently;

Correctness – (Already explained by the examples)

1. When all the good players start with the same value-line 9

2. If not all the good players get same bit value for majority vote - line 8 -> vote = 0, line 9 in next round

3. Else if, all the good players get same bit value for majority vote

   line 7 , line 8, line 9 next round

Expected running time

- Constant

- May not be terminated