Distinguishing sets of quantum states

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Abstract

Given two sets finite S_0 and S_1 of quantum states. We show necessary and sufficient conditions for distinguishing them by a measurement.

Let there be two finite sets of quantum states $S_0 = \{\rho_i : 1 \leq i \leq l\}$ and $S_1 = \{\sigma_i : 1 \leq i \leq l\}$. Please see [NC00] for a good introduction to information theory. By a *ϵ-separating measurement* T we mean a POVM element T such that $\forall \rho \in S_0$ and $\forall \sigma \in S_1$ we have $\mathsf{Tr} T\rho - \mathsf{Tr} T\sigma \geq \epsilon$ for some constant ϵ. For distributions μ_0 on S_0 and μ_1 on S_1, let $\rho_{\mu_0} = \mathsf{E}_{i \in \mu_0}[\rho_i]$ and $\sigma_{i \in \mu_1} = \mathsf{E}_{\mu_1}[\sigma_i]$. We show the following:

Theorem 0.1 *The following statements are equivalent:*

1. *There exists a ϵ-separating measurement T.*
2. *For all distributions μ_0 on S_0 and μ_1 on S_1, $\|\rho_{\mu_0} - \sigma_{\mu_1}\|_{\mathrm{tr}} \geq 2\epsilon$.*

For our proof we will need the following facts.

Fact 0.1 ([NC00]) *Given quantum states ρ, σ,*

$$\max_{T\text{: a POVM element}} \mathsf{Tr} T\rho - \mathsf{Tr} T\sigma = \frac{1}{2} \|\rho - \sigma\|_{\mathrm{tr}}$$

We have the following minimax theorem from game theory(see [OR94]).

Fact 0.2 *Let A_1, A_2 be non-empty, either finite or convex and compact subsets of $\mathbb{R}^n$. Let $u : A_1 \times A_2 \to \mathbb{R}$ be a continuous function. Let μ_1, μ_2 be distributions on A_1 and A_2 respectively. Then,*

$$\min_{\mu_1} \max_{a_2 \in A_2} E_{\mu_1}[u(a_1, a_2)] = \max_{\mu_2} \min_{a_1 \in A_1} E_{\mu_2}[u(a_1, a_2)]$$

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Proof:
1) $\Rightarrow$ 2) : If T is a separating measurement then from linearity of Tr operation we see that for all distributions μ_0 on S_0 and μ_1 on S_1, $\mathsf{Tr}T\rho_{\mu_0} - \mathsf{Tr}T\sigma_{\mu_1} \geq \epsilon$. Fact 0.1 now implies $\|\rho_{\mu_0} - \sigma_{\mu_1}\|_{\mathrm{tr}} \geq 2\epsilon$.

2) $\Rightarrow$ 1): Let us define sets $A_1 = \{T : T \text{ a POVM element}\}$ and $A_2 = \{(\rho,\sigma) : \rho \in S_0, \sigma \in S_1\}$. Let the function $u : A_1 \times A_2 \to \mathbb{R}$ be defined as $u(T,(\rho,\sigma)) = \mathsf{Tr}T\rho - \mathsf{Tr}T\sigma$. For a distribution μ on $S_0 \times S_1$, let μ_0 be the marginal distribution on S_0 and μ_1 be the marginal distribution on S_1. From fact 0.2 and the fact that a convex combination of POVM elements is also a POVM element and from linearity of Tr operation it follows:

$$\begin{aligned}
\max_T \min_{(\rho,\sigma)\in S_0\times S_1} \mathsf{Tr}T\rho - \mathsf{Tr}T\sigma &= \min_\mu \max_T \mathsf{E}_\mu[\mathsf{Tr}T\rho - \mathsf{Tr}T\sigma] \\
&= \min_\mu \max_T \mathsf{Tr}T\rho_{\mu_0} - \mathsf{Tr}T\sigma_{\mu_1} \\
&= \min_\mu \frac{1}{2}\|\rho_{\mu_0} - \sigma_{\mu_1}\|_{\mathrm{tr}} \text{ (from Fact 0.1)}
\end{aligned}$$

From above it is clear that 2) $\Rightarrow$ 1). ■

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References

[NC00] M. Nielsen and I. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[OR94] M. Osborne and A. Rubinstein. *A course in game theory*. MIT Press, 1994.