

# A new information-theoretic property about quantum states with an application to privacy in quantum communication\*

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## Abstract

We prove the following information-theoretic property about quantum states.

*Substate theorem:* Let  $\rho$  and  $\sigma$  be quantum states in the same Hilbert space with relative entropy  $S(\rho||\sigma) := \text{Tr} \rho(\log \rho - \log \sigma) = c$ . Then for all  $\epsilon > 0$ , there is a state  $\rho'$  such that the trace distance  $\|\rho' - \rho\|_{\text{tr}} := \text{Tr} \sqrt{(\rho' - \rho)^2} \leq \epsilon$ , and  $\rho'/2^{O(c/\epsilon^2)} \leq \sigma$ .

It states that if the relative entropy of  $\rho$  and  $\sigma$  is small, then there is a state  $\rho'$  close to  $\rho$ , i.e. with small trace distance  $\|\rho' - \rho\|_{\text{tr}}$ , that when scaled down by a factor  $2^{O(c)}$  ‘sits inside’, or becomes a ‘substate’ of,  $\sigma$ . This result has several applications in quantum communication complexity and cryptography. Using the substate theorem, we derive a privacy trade-off for the *set membership problem* in the two-party quantum communication model. Here Alice is given a subset  $A \subseteq [n]$ , Bob an input  $i \in [n]$ , and they need to determine if  $i \in A$ .

*Privacy trade-off for set membership:* In any two-party quantum communication protocol for the set membership problem, if Bob reveals only  $k$  bits of information about his input, then Alice must reveal at least  $n/2^{O(k)}$  bits of information about her input.

We also discuss relationships between various information theoretic quantities that arise naturally in the context of the substate theorem.

## 1 Introduction

The main contribution of this paper is a theorem, called the substate theorem; it states, roughly, that if the relative entropy,  $S(\rho||\sigma) := \text{Tr} \rho(\log \rho - \log \sigma)$ , of two quantum states  $\rho$  and  $\sigma$  is at most  $c$ , then there is a state  $\rho'$  close to  $\rho$  such that  $\rho'/2^{O(c)}$  *sits inside*  $\sigma$ . This implies that, as we will formalise later, state  $\sigma$  can ‘masquerade’ as state  $\rho$  with probability  $2^{-O(c)}$  in many situations. Before we discuss the substate theorem, let us first see a setting in which it is applied in order to get some motivation. This application concerns the trade-off in privacy in two-party quantum communication protocols for the set membership problem [MNSW98]. After that, we discuss the substate theorem proper followed by a brief description of several subsequent applications of the theorem.

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## 1.1 The set membership problem

**Definition 1** In the set membership problem  $\text{SetMemb}_n$ , Alice is given a subset  $A \subseteq [n]$  and Bob an element  $i \in [n]$ . The two parties are required to exchange messages according to a fixed protocol in order for the last recipient of a message to determine if  $i \in A$ . We often think of Alice's input as a string  $x \in \{0, 1\}^n$  which we view as the characteristic vector of the set  $A$ ; the protocol requires that in the end the last recipient output  $x_i$ . In this viewpoint, Bob's input  $i$  is called an index and the set membership problem is called the index function problem.

The set membership problem is a fundamental problem in communication complexity. In the classical setting, it was studied by Miltersen, Nisan, Safra and Wigderson [MNSW98], who showed that if Bob sends a total of at most  $b$  bits, then Alice must send  $n/2^{O(b)}$  bits. Note that this is optimal up to constants, as there is a trivial protocol where Bob sends the first  $b$  bits of his index to Alice, and Alice replies by sending the corresponding part of her bit string. The proof of Miltersen *et al.* relied on the *richness technique* they developed to analyse such protocols. However, here is a simple round-elimination argument that gives this lower bound, and as we will see below, this argument generalises to the quantum setting. Fix a protocol where Bob sends a total of at most  $b$  bits, perhaps spread over several rounds. We can assume without loss of generality that Bob is the last recipient of a message, otherwise we can augment the protocol by making Alice send the answer to Bob at the end which increases Alice's communication cost by one bit. Modify this protocol as follows. In the new protocol, Alice and Bob use shared randomness to guess all the messages of Bob. Alice sends her responses based on this guess. After this, if Bob finds that the guessed messages are exactly what he wanted to send anyway, he accepts the answer given by the original protocol; otherwise, he aborts the protocol. Thus, if the original protocol was correct with probability  $p$ , the new one-round protocol, when it does not abort, which happens with probability at least  $2^{-b}$ , is correct with probability at least  $p$ . A standard information theoretic argument now shows that in any such protocol Alice must send  $2^{-b} \cdot n(1 - H(p))$  bits (see e.g. [GKRdW06]; an argument as in [ANTV02] also gives a similar bound).

In the quantum setting, a special case of the set membership problem was studied by Ambainis, Nayak, Ta-Shma and Vazirani [ANTV02], where Bob is not allowed to send any message and there is no prior entanglement between Alice and Bob. They referred to this as *quantum random access codes*, because in this setting the problem can be thought of as Alice encoding  $n$  classical bits  $x$  using qubits in such a way that Bob is able to determine any one  $x_i$  with probability at least  $p \geq \frac{1}{2}$ . Note that in the quantum setting, unlike in its classical counterpart, it is conceivable that the measurement needed to determine  $x_i$  makes the state unsuitable for determining any of the other bits  $x_j$ . In fact, Ambainis *et al.* exhibit a quantum random access code encoding two classical bits  $(x_1, x_2)$  into one qubit such that any single bit  $x_i$  can be recovered with probability strictly greater than  $1/2$ , which is impossible classically. Their main result, however, was that any such quantum code must have  $n(1 - H(p))$  qubits (see also [Kla07, Theorem 5.8], for extension of this lower bound for entangled parties). They also gave a classical code with encoding length  $n(1 - H(p)) + O(\log n)$ , thus showing that quantum random access codes provide no substantial improvement over classical random access codes.

In this paper, we study the general set membership problem, where Alice and Bob are allowed to exchange quantum messages over several rounds as well as share prior entanglement. Ashwin Nayak (private communication) observed that the classical round elimination argument described above is applicable in the quantum setting: if Alice and Bob share prior entanglement in the form of EPR pairs, then using quantum teleportation [BBC<sup>+</sup>93], Bob's messages can be assumed to be classical. Now, Alice can guess Bob's messages, and we can combine the classical round elimination argument above with the results on random access codes to show that Alice must send at least  $2^{-(2b+1)} \cdot n(1 - H(p))$  qubits to Bob.

We strengthen these results and show that this trade-off between the communication required of Alice and Bob is in fact a trade-off in their privacy: if a protocol has the property that Bob ‘leaks’ only a small number of bits of *information* about his input, then in that protocol Alice must leak a large amount of information about her input; in particular, she must send a large number of qubits. Before we present our result, let us explain what we mean when we say that Bob leaks only a small number of bits of information about his input. Fix a protocol for set membership. Assume that Bob’s input  $J$  is a random element of  $[n]$ . Suppose Bob operates faithfully according to the protocol, but Alice deviates from it and manages to get her registers, say  $A$ , entangled with  $J$ : we say that Bob leaks only  $b$  bits of information about his input if the mutual information between  $J$  and  $A$ ,  $I(J : A)$ , is at most  $b$ . This must hold for all strategies adopted by Alice. Note that we do not assume that Bob’s messages contain only  $b$  qubits, they can be arbitrarily long. In the quantum setting, Alice has a big bag of tricks she can use in order to extract information from Bob. See Section 3.1 for an example of a cheating strategy for Alice, that exploits Alice’s ability to perform quantum operations. We show the following result.

**Result 1 (informal statement)** *If there is a quantum protocol for the set membership problem where Bob leaks only  $b$  bits of information about his input  $J$ , then Alice must leak  $\Omega(n/2^{O(b)})$  bits of information about her input  $x$ . In particular, this implies that Alice must send  $n/2^{O(b)}$  qubits.*

**Related work:** One can compare this with work on private information retrieval [CKGS98]. There, one requires that the party holding the database  $x$  know nothing about the index  $i$ . Nayak [Nay99] sketched an argument showing that in both classical and quantum settings, the party holding the database has to send  $\Omega(n)$  bits/qubits to the party holding the index. Result 1 generalises Nayak’s argument and shows a trade-off between the loss in privacy for the database user Bob, and the loss in privacy for the database server Alice.

Recently, Klauck [Kla02] studied privacy in quantum protocols. In Klauck’s setting, two players collaborate to compute a function, but at any point, one of the players might decide to terminate the protocol and try to infer something about the input of the other player using the bits in his possession. The players are *honest but curious*: in a sense, they don’t deviate from the protocol in any way other than, perhaps, by stopping early. In this model, Klauck shows that there is a protocol for the *set disjointness* function where neither player reveals more than  $O((\log n)^2)$  bits of information about his input, whereas in every classical protocol, at least one of the players leaks  $\Omega(\sqrt{n}/\log n)$  bits of information about his input. Our model of privacy is more stringent. We allow malicious players who can deviate arbitrarily from the protocol. An immediate corollary of our result is that for the set membership problem, one of the players must leak  $\Omega(\log n)$  bits of information. This implies a similar loss in privacy for several other problems, including the set disjointness problem.

**Privacy trade-off and the substate theorem:** We now briefly motivate the need for the substate theorem in showing the privacy trade-off in Result 1 above. We know from the communication trade-off argument for set membership presented above that in any protocol for the problem, if Bob sends only  $b$  qubits, then Alice must send  $n/2^{O(b)}$  qubits. Unfortunately, this argument is not applicable when the protocol does not promise that Bob sends only  $b$  qubits, but only ensures that the number of bits of information Bob leaks is at most  $b$ . So, the assumption is weaker. On the other hand, the conclusion now is stronger, for it asserts that Alice must leak  $n/2^{O(b)}$  bits of information, which implies that she must send at least these many qubits. The above argument relied on the fact that Alice could generate a distribution on messages, so that every potential message of Bob is well-represented in this distribution: if Bob’s messages are classical and  $b$  bits long, the uniform distribution is such a distribution—each  $b$  bit message appears in it with probability  $2^{-b}$ . Note that we are not assuming that messages of Bob have at most  $b$  qubits, so Alice cannot guess

these messages in this manner. Nevertheless, using only the assumption that Bob leaks at most  $b$  bits of information about his input, the substate theorem provides us an alternative for the uniform distribution. It allows us to prove the existence of a single quantum state that Alice and Bob can generate without access to Bob's input, after which if Bob is provided the input  $i$ , he can obtain the correct final state with probability at least  $2^{-O(b)}$  or abort if he cannot. After this, a standard quantum information theoretic argument implies that Alice must leak at least  $n/2^{O(b)}$  bits of information about her input (see e.g. [GKRdW06]; an argument as in [ANTV02] also gives a similar bound). The proof is discussed in detail in Section 3.

## 1.2 The substate theorem

It will be helpful to first consider the classical analogue of the substate theorem. Let  $P$  and  $Q$  be probability distributions on the set  $[n]$  such that their relative entropy is bounded by  $c$ , that is

$$S(P\|Q) := \sum_{i \in [n]} P(i) \log_2 \frac{P(i)}{Q(i)} \leq c \quad (1)$$

When  $c$  is small, this implies that  $P$  and  $Q$  are close to each other in *total variation distance*; indeed, one can show that (see e.g. [CT91, Lemma 12.6.1])

$$\|P - Q\|_1 := \sum_{i \in [n]} |P(i) - Q(i)| \leq \sqrt{(2 \ln 2)c}. \quad (2)$$

That is, the probability of an event  $\mathcal{E} \subseteq [n]$  in  $P$  is close to its probability in  $Q$ :  $|P(\mathcal{E}) - Q(\mathcal{E})| \leq \sqrt{(c \ln 2)/2}$ . Now consider the situation when  $c \gg 1$ . In that case, expression (2) becomes weak, and it is not hard to construct examples where  $\|P - Q\|_1$  is very close to 2. Thus by bounding  $\|P - Q\|_1$  alone, we cannot infer that an event  $\mathcal{E}$  with probability  $3/4$  in  $P$  has any non-zero probability in  $Q$ . But is it true that when  $S(P\|Q) < +\infty$  and  $P(\mathcal{E}) > 0$ , then  $Q(\mathcal{E}) > 0$ ? Yes! To see this, let us reinterpret the expression in (1) as the expectation of  $\log P(i)/Q(i)$  as  $i$  is chosen according to  $P$ . Thus, one is lead to believe that if  $S(P\|Q) \leq c < +\infty$ , then  $\log P(i)/Q(i)$  is typically bounded by  $c$ , that is,  $P(i)/Q(i)$  is typically bounded by  $2^c$ . One can formalise this intuition and show, for all  $r \geq 1$ ,

$$\Pr_{i \leftarrow P} \left[ \frac{P(i)}{Q(i)} > 2^{r(c+1)} \right] < \frac{1}{r}. \quad (3)$$

We now briefly sketch a proof of the above inequality. Note that since some terms involved in the relative entropy expression (Eq. (1)) could be negative, we cannot apply the Markov's inequality immediately. Let  $\text{Good} := \{i : P(i)/2^{r(c+1)} \leq Q(i)\}$ ,  $\text{Bad} := [n] \setminus \text{Good}$ . By concavity of the logarithm function, we get

$$P(\text{Good}) \log \frac{P(\text{Good})}{Q(\text{Good})} + P(\text{Bad}) \log \frac{P(\text{Bad})}{Q(\text{Bad})} \leq S(P\|Q) \leq c.$$

By elementary calculus,  $P(\text{Good}) \log \frac{P(\text{Good})}{Q(\text{Good})} > -1$ . Thus we get  $P(\text{Bad}) \cdot r(c+1) < c+1$ , proving the above inequality.

We now define a new probability distribution  $P'$  as follows:

$$P'(i) := \begin{cases} \frac{P(i)}{P(\text{Good})} & i \in \text{Good} \\ 0 & i \in \text{Bad} \end{cases},$$

that is, in  $P'$  we just discard the bad values of  $i$  and renormalise. Now,  $\frac{r-1}{r2^{r(c+1)}}P'$  is dominated by  $Q$  everywhere. We have thus shown the classical analogue of the desired substate theorem.

**Result 2' (Classical substate theorem)** *Let  $P, Q$  be probability distributions on the same sample space with  $S(P\|Q) \leq c$ . Then for all  $r > 1$ , there exist distributions  $P', P''$  such that*

$$\|P - P'\|_1 \leq \frac{2}{r} \quad \text{and} \quad Q = \alpha P' + (1 - \alpha)P'',$$

where  $\alpha := \frac{r-1}{r2^{r(c+1)}}$ .

Let us return to our event  $\mathcal{E}$  that occurred with some small probability  $p$  in  $P$ . Now, if we take  $r$  to be  $2/p$ , then  $\mathcal{E}$  occurs with probability at least  $p/2$  in  $P'$ , and hence appears with probability  $p/2^{O(c/p)}$  in  $Q$ . Thus, we have shown that even though  $P$  and  $Q$  are far apart as distributions, events that have positive probability, no matter how small, in  $P$ , continue to have positive probability in  $Q$ .

The main contribution of this paper is a quantum analogue of Result 2'. To state it, we recall that the relative entropy of two quantum states  $\rho, \sigma$  in the same Hilbert space is defined as  $S(\rho\|\sigma) := \text{Tr} \rho(\log \rho - \log \sigma)$ , and the *trace distance* between them is defined as  $\|\rho - \rho'\|_{\text{tr}} := \text{Tr} \sqrt{(\rho - \rho')^2}$ .

**Result 2 (Quantum substate theorem)** *Suppose  $\rho$  and  $\sigma$  are quantum states in the same Hilbert space with  $S(\rho\|\sigma) \leq c$ . Then for all  $r > 1$ , there exist states  $\rho', \rho''$  such that*

$$\|\rho - \rho'\|_{\text{tr}} \leq \frac{2}{\sqrt{r}} \quad \text{and} \quad \sigma = \alpha \rho' + (1 - \alpha)\rho'',$$

where  $\alpha := \frac{r-1}{r2^{r c'}} \quad \text{and} \quad c' := c + 4\sqrt{c+2} + 2\log(c+2) + 6$ .

The quantum substate theorem has been stated above in a form that brings out the analogy with the classical statement in Result 2'. In Section 4, we have a more nuanced statement which is often better suited for applications.

**Remark:** Using the quantum substate theorem and arguing as above, one can conclude that if an event  $\mathcal{E}$  has probability  $p$  in  $\rho$ , then its probability  $q$  in  $\sigma$  is at least  $q \geq \frac{p}{2^{O(c/p^2)}}$ ,  $c = S(\rho\|\sigma)$ . Actually, one can show the stronger result that  $q \geq \frac{p}{2^{O(c/p)}}$  as follows. Using the fact that relative entropy cannot increase after doing a measurement, we get

$$p \log \frac{p}{q} + (1 - p) \log \frac{1 - p}{1 - q} \leq S(\rho\|\sigma) \leq c.$$

We now argue as in the proof of Result 2' to show the stronger lower bound on  $q$ .

In view of this, one may wonder if there is any motivation at all in proving a quantum substate theorem. Recall however, that the quantum substate theorem gives a structural relationship between  $\rho$  and  $\sigma$  which is useful in many applications e.g. privacy trade-off for set membership discussed earlier. It does not seem possible in these applications to replace this structural relationship by considerations about the relative probabilities of an event  $\mathcal{E}$  in  $\rho$  and  $\sigma$ . In our privacy trade-off application,  $\sigma$  plays the role of the state that Alice and Bob can generate without access to Bob's input, and  $\rho$  plays the role of the correct final state of Bob in the protocol. To prove the trade-off,  $\sigma$  should be able to 'masquerade' as  $\rho$  with probability  $2^{-O(b)}$ ,  $b$  being the amount of information Bob leaks about his input. Also, Bob should *know* whether the 'masquerade' succeeded or not so that he can abort if it fails, and it is this requirement that needs the substate property.

The ideas used to arrive at Result 2' do not immediately generalise to prove Result 2, because  $\rho$  and  $\sigma$  need not be simultaneously diagonalisable. As it turns out, our proof of the quantum substate theorem takes an indirect route. First, by exploiting the Fuchs and Caves [FC95] characterisation of fidelity and a

minimax theorem of game theory, we obtain a ‘lifting’ theorem about an ‘observational’ version of relative entropy; this statement is interesting on its own. Using this ‘lifting’ theorem, and a connection between the ‘observational’ version of relative entropy and actual relative entropy, we argue that it is enough to verify the original statement when  $\rho$  and  $\sigma$  reside in a two-dimensional space and  $\rho$  is a pure state. The two dimensional case is then established by a direct computation.

### 1.3 Other applications of the substate theorem

The conference version of this paper [JRS02], in which the substate theorem was first announced, described two applications of the theorem. The first application provided tight privacy trade-offs for the set membership problem, which we have discussed above. This application is a good illustration of the use of the substate theorem, for several applications have the same structure. The second application showed tight lower bounds for the *pointer chasing problem* [NW93, KNTZ07], thereby establishing that the lower bounds shown by Ponzio, Radhakrishnan and Venkatesh [PRV01] in the classical setting are valid also for quantum protocols without prior entanglement.

Subsequent to [JRS02], several applications of the classical and quantum substate theorems have been discovered. We briefly describe these results now. Earlier, in related but independent work Chakrabarti, Shi, Wirth and Yao [CSWY01] discovered their very influential *information cost* approach for obtaining *direct sum* results in communication complexity. Jain, Radhakrishnan and Sen [JRS03] observed that the results of Chakrabarti *et al.* could be derived more systematically using the classical substate theorem. This approach allowed them to extend Chakrabarti *et al.*’s direct sum results, which applied to one-round and simultaneous message protocols under the uniform distribution on inputs, to two-party multiple round protocols under all product distributions on inputs. Ideas from [JRS03] were then applied by Chakrabarti and Regev [CR04] to obtain their tight lower bound on data structures for the *approximate nearest neighbour problem* on the Hamming cube. The classical substate theorem has also been used by Jain, Klauck and Nayak [JKN08] in order to prove direct product theorems for classical communication complexity.

The quantum substate theorem, the main result of this paper, has also found several other applications. Jain, Radhakrishnan and Sen [JRS05] used it to show how any two-party multiple round quantum protocol for computing any relation, where Alice leaks only  $a$  bits of information about her input and Bob leaks only  $b$  bits of information about his, can be transformed to a one-round quantum protocol with prior entanglement where Alice transmits just  $a2^{O(b)}$  bits to Bob. Note that plain Schumacher compression [Sch95] cannot be used to prove such a result, since we require a ‘one-shot’ as opposed to an asymptotic result. Also there can be interaction in a general communication protocol, as well it could be that the reduced state of any single party can be mixed. Jain *et al.*’s compression result gives an alternative proof of Result 1, because the work of Ambainis *et al.* [ANTV02] implies that in any such protocol for set membership Alice must send  $\Omega(n)$  bits to Bob. Jain *et al.* also used the classical and quantum substate theorems to prove worst case direct sum results for simultaneous message and one round classical and quantum protocols, improving on [JRS03]. More recently, using the quantum substate theorem Jain [Jai06] obtained a nearly tight characterisation of the communication complexity of *remote state preparation*, an area that has received considerable attention lately. The substate theorem has also found application in the study of quantum cryptographic protocols: using it, Jain [Jai08] showed nearly tight bounds on the *binding-concealing* trade-offs for *quantum string commitment* schemes.

## 1.4 Organisation of the rest of the paper

In the next section, we recall some basic facts from classical and quantum information theory that will be used in the rest of the paper. In Section 3, we formally define our model of privacy loss in quantum communication protocols and prove our privacy trade-off result for set membership assuming the substate theorem. In Section 4, we give the actual statement of the substate theorem that is used in our privacy trade-offs, and a complete proof for it. Sections 3 and 4 may be read independently of each other. In Section 5 we mention some open problems, and finally in the appendix we discuss relationships between various information theoretic quantities that arise naturally in the context of the substate theorem. The appendix may be read independently of Section 3.

## 2 Information theory background

We now recall some basic definitions and facts from classical and quantum information theory, which will be useful later. For excellent introductions to classical and quantum information theory, see the books by Cover and Thomas [CT91] and Nielsen and Chuang [NC00] respectively.

In this paper, all functions will have finite domains and ranges, all sample spaces will be finite, all random variables will have finite range and all Hilbert spaces are over complex numbers and finite dimensional. All logarithms are taken to base two. We will use the notation  $A \geq B$  for operators  $A, B$  in the same Hilbert space  $\mathcal{H}$  as a shorthand for the statement ‘ $A - B$  is positive semidefinite’. Thus,  $A \geq 0$  denotes that  $A$  is positive semidefinite and  $A > 0$  denotes that  $A$  is (strictly) positive definite. For a positive semidefinite operator  $A$ ,  $\sqrt{A}$  denotes the unique positive semidefinite operator such that  $\sqrt{A} \cdot \sqrt{A} = A$ .

We start off by recalling the definition of a quantum state.

**Definition 2 (Quantum state)** *A quantum state or a density matrix in a Hilbert space  $\mathcal{H}$  is a positive semidefinite operator  $\rho$  on  $\mathcal{H}$  with unit trace.*

Note that a classical probability distribution can be thought of as a special case of a quantum state with diagonal density matrix. An important class of quantum states are what are known as *pure* states, which are states of the form  $|\psi\rangle\langle\psi|$ , where  $|\psi\rangle$  is a unit vector in  $\mathcal{H}$ . Often, we abuse notation and refer to  $|\psi\rangle$  itself as the pure quantum state; note that this notation is ambiguous up to multiplication by a unit complex number. The *support* of a density matrix  $\rho$ , denoted by  $\text{supp}(\rho)$ , is the span of eigenvectors corresponding to non-zero eigenvalues of  $\rho$ .

Let  $\mathcal{H}, \mathcal{K}$  be two Hilbert spaces and  $\omega$  a quantum state in the *bipartite system*  $\mathcal{H} \otimes \mathcal{K}$ . The *reduced* quantum state of  $\mathcal{H}$  is given by *tracing out*  $\mathcal{K}$ , also known as the *partial trace*  $\text{Tr}_{\mathcal{K}} \omega := \sum_k (\mathbb{1}_{\mathcal{H}} \otimes \langle k |) \omega (\mathbb{1}_{\mathcal{H}} \otimes |k\rangle)$  where  $\mathbb{1}_{\mathcal{H}}$  is the identity operator on  $\mathcal{H}$  and the summation is over an orthonormal basis for  $\mathcal{K}$ . It is easy to see that the partial trace is independent of the choice of the orthonormal basis for  $\mathcal{K}$ . For a quantum state  $\rho$  in  $\mathcal{H}$ , any quantum state  $\omega$  in  $\mathcal{H} \otimes \mathcal{K}$  such that  $\text{Tr}_{\mathcal{K}} \omega = \rho$  is said to be an *extension* of  $\rho$  in  $\mathcal{H} \otimes \mathcal{K}$ ; if  $\omega$  is pure, it is said, more specifically, to be a *purification*.

We next define a POVM element, which formalises the notion of a single outcome of a general measurement on a quantum state.

**Definition 3 (POVM element)** *A POVM (positive operator valued measure) element  $F$  on Hilbert space  $\mathcal{H}$  is a positive semidefinite operator on  $\mathcal{H}$  such that  $F \leq \mathbb{1}$ , where  $\mathbb{1}$  is the identity operator on  $\mathcal{H}$ .*

We now define a POVM which represents the most general form of a measurement allowed by quantum mechanics.

**Definition 4 (POVM)** A POVM  $\mathcal{F}$  on Hilbert space  $\mathcal{H}$  is a finite set of POVM elements  $\{F_1, \dots, F_k\}$  on  $\mathcal{H}$  such that  $\sum_{i=1}^k F_i = \mathbb{1}$ , where  $\mathbb{1}$  is the identity operator on  $\mathcal{H}$ .

When a quantum state  $\rho \in H$  is measured according to the POVM  $\{F_i : i \in [k]\}$ , the probability that outcome  $i \in [k]$  is observed is  $\text{Tr}(F_i \rho)$ . For a quantum state  $\rho$  in  $\mathcal{H}$ , let  $\mathcal{F}\rho$  denote the probability distribution  $\{p_1, \dots, p_k\}$  on  $[k]$ , where  $p_i := \text{Tr}(F_i \rho)$ .

Typically, the distance between two probability distributions  $P, Q$  on the same sample space  $\Omega$  is measured in terms of the *total variation distance* defined as  $\|P - Q\|_1 := \sum_{i \in \Omega} |P(i) - Q(i)|$ . The quantum analogue of the total variation distance is known as the *trace distance*.

**Definition 5 (Trace distance)** Let  $\rho, \sigma$  be quantum states in the same Hilbert space. Their trace distance is defined as  $\|\rho - \sigma\|_{\text{tr}} := \text{Tr} \sqrt{(\rho - \sigma)^2}$ .

If we think of probability distributions as diagonal density matrices, then the trace distance between them is nothing but their total variation distance. For pure states  $|\psi\rangle, |\phi\rangle$  it is easy to see that their trace distance is given by  $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\text{tr}} = 2\sqrt{1 - |\langle\psi|\phi\rangle|^2}$ . The following fundamental fact shows that the trace distance between two density matrices bounds how well one can distinguish between them by a POVM. A proof can be found in [Hel76].

**Fact 1** Let  $\rho, \sigma$  be density matrices in the same Hilbert space  $\mathcal{H}$ . Let  $\mathcal{F}$  be a POVM on  $\mathcal{H}$ . Then,  $\|\mathcal{F}\rho - \mathcal{F}\sigma\|_1 \leq \|\rho - \sigma\|_{\text{tr}}$ . Also, there is a two-outcome orthogonal measurement that achieves equality above.

Another measure of distinguishability between two probability distributions  $P, Q$  on the same sample space  $\Omega$  is the *Bhattacharya distinguishability coefficient* defined as  $B(P, Q) := \sum_{i \in \Omega} \sqrt{P(i)Q(i)}$ . Its quantum analogue is known as *fidelity*. We will need several facts about fidelity in order to prove the quantum substate theorem.

**Definition 6 (Fidelity)** Let  $\rho, \sigma$  be density matrices in the same Hilbert space  $\mathcal{H}$ . Their fidelity is defined as  $B(\rho, \sigma) := \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}$ .

The fidelity, or sometimes its square, is also referred to as the “transition probability” of Uhlmann. For probability distributions, the fidelity turns out to be the same as their Bhattacharya distinguishability coefficient. Jozsa [Joz94] gave an elementary proof for finite dimensional Hilbert spaces of the following basic and remarkable property about fidelity (also see [Uhl76]).

**Fact 2** Let  $\rho, \sigma$  be density matrices in the same Hilbert space  $\mathcal{H}$ . Then,  $B(\rho, \sigma) = \sup_{\mathcal{K}, |\psi\rangle, |\phi\rangle} |\langle\psi|\phi\rangle|$ , where  $\mathcal{K}$  ranges over all Hilbert spaces and  $|\psi\rangle, |\phi\rangle$  range over all purifications of  $\rho, \sigma$  respectively in  $\mathcal{H} \otimes \mathcal{K}$ . Also, for any Hilbert space  $\mathcal{K}$  such that  $\dim(\mathcal{K}) \geq \dim(\mathcal{H})$ , there exist purifications  $|\psi\rangle, |\phi\rangle$  of  $\rho, \sigma$  in  $\mathcal{H} \otimes \mathcal{K}$ , such that  $B(\rho, \sigma) = |\langle\psi|\phi\rangle|$ .

We will also need the following fact about fidelity, proved by Fuchs and Caves [FC95].

**Fact 3** Let  $\rho, \sigma$  be density matrices in the same Hilbert space  $\mathcal{H}$ . Then  $B(\rho, \sigma) = \inf_{\mathcal{F}} B(\mathcal{F}\rho, \mathcal{F}\sigma)$ , where  $\mathcal{F}$  ranges over POVMs on  $\mathcal{H}$ . In fact, the infimum above can be attained by a complete orthogonal measurement on  $\mathcal{H}$ .

The most general operation on a density matrix allowed by quantum mechanics is what is called a *completely positive trace preserving superoperator*, or superoperator for short. Let  $\mathcal{H}, \mathcal{K}$  be Hilbert spaces. A superoperator  $\mathcal{T}$  from  $\mathcal{H}$  to  $\mathcal{K}$  maps quantum states  $\rho$  in  $\mathcal{H}$  to quantum states  $\mathcal{T}\rho$  in  $\mathcal{K}$ , and is described

by a finite collection of linear maps  $\{A_1, \dots, A_l\}$  from  $\mathcal{H}$  to  $\mathcal{K}$  called *Kraus operators* such that,  $\mathcal{T}\rho = \sum_{i=1}^l A_i \rho A_i^\dagger$ . Unitary transformations, taking partial traces and POVMs are special cases of superoperators.

Let  $X$  be a classical random variable. Let  $P$  denote the probability distribution induced by  $X$  on its range  $\Omega$ . The *Shannon entropy* of  $X$  is defined as  $H(X) := H(P) := -\sum_{i \in \Omega} P(i) \log P(i)$ . For any  $0 \leq p \leq 1$ , the *binary entropy* of  $p$  is defined as  $H(p) := H((p, 1-p)) = -p \log p - (1-p) \log(1-p)$ . If  $A$  is a quantum system with density matrix  $\rho$ , then its *von Neumann entropy*  $S(A) := S(\rho) := -\text{Tr} \rho \log \rho$ . It is obvious that the von Neumann entropy of a probability distribution equals its Shannon entropy. If  $A, B$  are two disjoint quantum systems, the *mutual information* of  $A$  and  $B$  is defined as  $I(A : B) := S(A) + S(B) - S(AB)$ ; mutual information of two random variables is defined analogously. By a *quantum encoding*  $M$  of a classical random variable  $X$  on  $m$  qubits, we mean that there is a bipartite quantum system with joint density matrix  $\sum_x \text{Pr}[X = x] \cdot |x\rangle\langle x| \otimes \rho_x$ , where the first system is the random variable, the second system is the quantum encoding and an  $x$  in the range of  $X$  is encoded by a quantum state  $\rho_x$  on  $m$  qubits. The reduced state of the first system is nothing but the probability distribution  $\sum_x \text{Pr}[X = x] \cdot |x\rangle\langle x|$  on the range of  $X$ . The reduced state of the second system is the *average code word*  $\rho := \sum_x \text{Pr}[X = x] \cdot \rho_x$ . The mutual information of this encoding is given by  $I(X : M) = S(X) + S(M) - S(XM) = S(\rho) - \sum_x \text{Pr}[X = x] \cdot S(\rho_x)$ .

We now define the *relative entropy* of a pair of quantum states.

**Definition 7 (Relative entropy)** *If  $\rho, \sigma$  are quantum states in the same Hilbert space, their relative entropy is defined as  $S(\rho||\sigma) := \text{Tr}(\rho(\log \rho - \log \sigma))$ .*

For probability distributions  $P, Q$  on the same sample space  $\Omega$ , the above definition reduces to  $S(P||Q) = \sum_{i \in \Omega} P(i) \log \frac{P(i)}{Q(i)}$ . The following fact lists some useful properties of relative entropy. Proofs can be found in [NC00, Chapter 11]. The monotonicity property below is also called *Lindblad-Uhlmann monotonicity*.

**Fact 4** *Let  $\rho, \sigma$  be density matrices in the same Hilbert space  $\mathcal{H}$ . Then,*

1.  $S(\rho||\sigma) \geq 0$ , with equality iff  $\rho = \sigma$ ;
2.  $S(\rho||\sigma) < +\infty$  iff  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ ;
3.  $D(\cdot||\cdot)$  is continuous in its two arguments over the domain of pairs of states  $(\rho, \sigma)$  such that  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ ;
4. (Unitary invariance) *If  $U$  is a unitary transformation on  $\mathcal{H}$ ,  $S(U\rho U^\dagger||U\sigma U^\dagger) = S(\rho||\sigma)$ .*
5. (Monotonicity) *Let  $\mathcal{L}$  be a Hilbert space and  $\mathcal{T}$  be a completely positive trace preserving superoperator from  $\mathcal{H}$  to  $\mathcal{L}$ . Then,  $S(\mathcal{T}\rho||\mathcal{T}\sigma) \leq S(\rho||\sigma)$ .*

The following fact relates mutual information to relative entropy, and is easy to prove.

**Fact 5** *Let  $X$  be a classical random variable and  $M$  be a quantum encoding of  $X$  i.e. each  $x$  in the range of  $X$  is encoded by a quantum state  $\rho_x$ . Let  $\rho := \sum_x \text{Pr}[X = x] \cdot \rho_x$  be the average code word. Then,  $I(X : M) = \sum_x \text{Pr}[X = x] \cdot S(\rho_x||\rho)$ .*

The next fact is an extension of the random access code arguments of [ANTV02], and was proved by Gavinsky, Kempe, Regev and de Wolf [GKRdW06, Lemma 1].

**Fact 6** *Let  $X = X_1 \cdots X_n$  be a classical random variable of  $n$  uniformly distributed bits. Let  $M$  be a quantum encoding of  $X$  on  $m$  qubits. For each  $i \in [n]$ , suppose there is a POVM  $\mathcal{F}_i$  on  $M$  with three outcomes  $0, 1, ?$ . Let  $Y_i$  denote the random variable obtained by applying  $F_i$  to  $M$ . Suppose there are real*

numbers  $0 \leq \lambda_i, \epsilon_i \leq 1$  such that  $\Pr[Y_i \neq ?] \geq \lambda_i$  and  $\Pr[Y_i = X_i \mid Y_i \neq ?] \geq 1/2 + \epsilon_i$ , where the probability arises from the randomness in  $X$  as well as the randomness of the outcome of  $\mathcal{F}_i$ . Then,

$$\sum_{i=1}^n \lambda_i \epsilon_i^2 \leq \sum_{i=1}^n \lambda_i (1 - H(1/2 + \epsilon_i)) \leq I(X : M) \leq m.$$

### 3 Privacy trade-offs for set membership

In this section, we prove a trade-off between privacy loss of Alice and privacy loss of Bob for the set membership problem  $\text{SetMemb}_n$  assuming the substate theorem. We then embed index function into other functions using the concept of VC-dimension and show privacy trade-offs for some other problems. But first, we formally define our model of privacy loss in quantum communication protocols.

#### 3.1 Quantum communication protocols

We consider two party quantum communication protocols as defined by Yao [Yao93]. Let  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$  be sets and  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$  be a function. There are two players Alice and Bob, who hold qubits. Alice gets an input  $x \in \mathcal{X}$  and Bob an input  $y \in \mathcal{Y}$ . When the communication protocol  $\mathcal{P}$  starts, Alice and Bob each hold some ‘work qubits’ initialised to the all-zeroes state. Alice and Bob may also share an input independent state  $|\psi\rangle$ , providing ‘prior entanglement’. Thus, the initial superposition is simply  $|0\rangle|\psi\rangle|0\rangle$ , where the first and last registers denote the work qubits of Alice and Bob respectively. Some of the qubits of  $|\psi\rangle$  belong to Alice, the rest belong to Bob. The players take turns to communicate to compute  $f(x, y)$ . Suppose it is Alice’s turn. Alice makes a unitary transformation on the qubits in her possession depending on  $x$  only, and then send some qubits to Bob. Sending qubits does not change the overall superposition, but rather the ownership of the qubits. This allows Bob to apply his next unitary transformation, which depends on  $y$  only, on his original qubits plus the newly received qubits. At the end of the protocol  $\mathcal{P}$ , some qubits belonging to the last recipient, called the ‘answer qubits’, are measured in the computational basis so as to output an answer  $\mathcal{P}(x, y)$ . For each  $(x, y) \in \mathcal{X} \times \mathcal{Y}$  the unitary transformations that are applied, as well as the qubits that are to be sent in each round, the number of rounds, the choice of the starting player, and the designation of which qubits are to be treated as ‘answer qubits’ are specified in advance by the protocol  $\mathcal{P}$ . We say that  $\mathcal{P}$  computes  $f$  with  $\epsilon$ -error in the worst case, if  $\max_{x,y} \Pr[\mathcal{P}(x, y) \neq f(x, y)] \leq \epsilon$ . We say that  $\mathcal{P}$  computes  $f$  with  $\epsilon$ -error with respect to a probability distribution  $\mu$  on  $\mathcal{X} \times \mathcal{Y}$ , if  $\mathbb{E}_\mu[\Pr[\mathcal{P}(x, y) \neq f(x, y)]] \leq \epsilon$ . The communication complexity of  $\mathcal{P}$  is defined to be the total number of qubits exchanged. Note that seemingly more general models of communication protocols can be thought of where superoperators may be applied by the parties instead of unitary transformations, and arbitrary POVM may be used in order to output the answer of the protocol instead of measuring in the computational basis. But such models can be converted to the unitary model above without changing the error probabilities, communication complexity, and as we will see later, privacy loss to a cheating party.

Given a probability distribution  $\mu$  on  $\mathcal{X} \times \mathcal{Y}$  we define  $|\mu\rangle := \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \sqrt{\mu(x,y)} |x\rangle|y\rangle$ . Running protocol  $\mathcal{P}$  with superposition  $|\mu\rangle$  fed to Alice’s and Bob’s inputs means that we first create the state  $\sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \sqrt{\mu(x,y)} |x\rangle|0\rangle|\psi\rangle|0\rangle|y\rangle$ , then feed the middle three registers to  $\mathcal{P}$  and let  $\mathcal{P}$  run its course. We define the success probability of  $\mathcal{P}$  when  $|\mu\rangle$  is fed to Alice’s and Bob’s inputs to be the probability that measuring the inputs and the answer qubits in the computational basis at the end of  $\mathcal{P}$  produces consistent results. Running protocol  $\mathcal{P}$  with mixture  $\mu$  fed to Alice’s and Bob’s inputs means that we first create the state  $\sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \mu(x,y) |x\rangle\langle x||0\rangle\langle 0||\psi\rangle\langle\psi||0\rangle\langle 0||y\rangle\langle y|$ , then feed the middle three registers to  $\mathcal{P}$  and let  $\mathcal{P}$  run its course. The success probability of  $\mathcal{P}$  when mixture  $\mu$  is fed to Alice’s and Bob’s inputs is defined

analogously. It is easy to see that the success probability of  $\mathcal{P}$  on superposition  $|\mu\rangle$  is the same as the success probability on mixture  $\mu$ , and both are equal to  $E_\mu[\Pr[\mathcal{P}(x, y) = f(x, y)]]$ .

Now let  $\mu_X, \mu_Y$  be probability distributions on  $\mathcal{X}, \mathcal{Y}$ , and let  $\mu := \mu_X \times \mu_Y$  denote the product distribution on  $\mathcal{X} \times \mathcal{Y}$ . Let  $\mathcal{P}$  be the prescribed *honest* protocol for  $f$ . In the ‘honest’ run of  $\mathcal{P}$  mixture  $\mu$  is fed to Alice’s and Bob’s inputs. Now let us suppose that Bob turns ‘malicious’ and deviates from the prescribed protocol  $\mathcal{P}$  in order to learn as much as he can about Alice’s input. Note that Alice remains honest in this scenario i.e. she continues to follow  $\mathcal{P}$ . Thus, Alice and Bob are now actually running a ‘cheating’ protocol  $\tilde{\mathcal{P}}$ . Let registers  $A, X, B, Y$  denote Alice’s work qubits, Alice’s input qubits, Bob’s work qubits and Bob’s input qubits respectively at the end of  $\tilde{\mathcal{P}}$ . The *privacy leakage* from Alice to Bob in  $\tilde{\mathcal{P}}$  is captured by the mutual information  $I_{\tilde{\mathcal{P}}}(X : BY)$  between Alice’s input register  $X$  and Bob’s registers  $BY$  at the end of protocol  $\tilde{\mathcal{P}}$ . The notation  $I_{\tilde{\mathcal{P}}}(\cdot : \cdot)$  is just used to emphasise the fact that we are referring to the standard mutual information between two quantum systems at the end of protocol  $\tilde{\mathcal{P}}$ . Note that the state of register  $X$  is still the mixture  $\mu_X$ . We want to study how large  $\sup I_{\tilde{\mathcal{P}}}(X : BY)$  can be for a given function  $f$ , product distribution  $\mu$ , and protocol  $\mathcal{P}$ , where the supremum is taken over all ‘cheating’ protocols  $\tilde{\mathcal{P}}$  wherein Bob can be arbitrarily malicious but Alice continues to follow  $\mathcal{P}$  honestly. We shall call this quantity the *privacy loss* of  $\mathcal{P}$  from Alice to Bob. Privacy leakage and privacy loss from Bob to Alice can be defined similarly.

One of the ways that Bob can cheat (even without Alice realising it!) is by running  $\mathcal{P}$  with the superposition  $|\mu_Y\rangle := \sum_{y \in \mathcal{Y}} \sqrt{\mu_Y(y)} |y\rangle$  fed to register  $Y$ . Alice remains honest and follows  $\mathcal{P}$ ; the state of her input register  $X$  continues to be the mixture  $\mu_X$ . This method of cheating gives Bob at least as much information about Alice’s input as in the ‘honest’ run of  $\mathcal{P}$  when the mixture  $\mu_Y$  is fed to  $Y$ . Sometimes it can give much more. Consider the set membership problem, where Alice has a bit string  $x$  which denotes the characteristic vector of a subset of  $[n]$  and Bob has an  $i \in [n]$ . Consider a *clean* protocol  $\mathcal{P}$  for the index function problem. Recall that a protocol  $\mathcal{P}$  is said to be clean if the work qubits of both the players except the answer qubits are in the state  $|0\rangle$  at the end of  $\mathcal{P}$ . We shall show a privacy trade-off result for  $\mathcal{P}$  under the uniform distribution on the inputs of the two players. For simplicity, assume that  $\mathcal{P}$  is errorless (an error of  $1/4$  will only change the privacy losses by a multiplicative constant). Suppose Alice cheats by feeding a uniform superposition over bit strings into her input register  $X$ , and then running  $\mathcal{P}$ . Bob is honest, and has a random  $i \in [n]$ . At the end of this ‘cheating’ run of  $\mathcal{P}$ , Alice applies a Hadamard transformation on each of the registers  $X_j, 1 \leq j \leq n$ . Suppose she were to measure them now in the computational basis. For all  $j \neq i$ , she would measure  $|0\rangle$  with probability 1. For  $j = i$ , she would measure 1 with probability  $1/2$ . Thus, Alice has extracted about  $\log n/2$  bits of information about Bob’s index  $i$ . An ‘honest’ run of  $\mathcal{P}$  would have yielded Alice only 1 bit of information about  $i$ .

We now define a *superpositional privacy loss* inspired by the above example. We consider the ‘cheating’ run of  $\mathcal{P}$  wherein mixture  $\mu_X$  is fed to register  $X$  and superposition  $|\mu_Y\rangle$  to register  $Y$ . Denote this ‘cheating’ run by  $\mathcal{P}'$ . Let  $I'(X : BY)$  denote the mutual information of Alice’s input register  $X$  with Bob’s registers  $BY$  at the end of protocol  $\mathcal{P}'$ . The notation  $I'(\cdot : \cdot)$  is just used to emphasise the fact that we are referring to the standard mutual information between two quantum systems at the end of protocol  $\mathcal{P}'$ .

**Definition 8 (Superpositional privacy loss)** *The superpositional privacy loss of  $\mathcal{P}$  for function  $f$  on the product distribution  $\mu$  from Alice to Bob is defined as  $L^{\mathcal{P}}(f, \mu, A, B) := I'(X : BY)$ . The superpositional privacy loss from Bob to Alice,  $L^{\mathcal{P}}(f, \mu, B, A)$ , is defined similarly. The superpositional privacy loss of  $\mathcal{P}$  for  $f$ ,  $L^{\mathcal{P}}(f)$ , is the maximum over all product distributions  $\mu$ , of  $\max\{L^{\mathcal{P}}(f, \mu, A, B), L^{\mathcal{P}}(f, \mu, B, A)\}$ .*

**Remarks:**

1. Our notion of superpositional privacy loss can be viewed as a quantum analogue of the ‘combinatorial-informational’ bounded error measure of privacy loss,  $I_{c-i}^*$ , in Bar-Yehuda et al. [BCKO93].

2. Klauck [Kla02], based on Cleve et al. [CvDNT98], has made a similar observation about  $\Omega(n)$  privacy loss for clean protocols computing the inner product mod 2 function. The significance of our lower bounds on privacy loss is that they make *no assumptions* about the protocol  $\mathcal{P}$ .
3. In the definition of privacy loss by Klauck [Kla02], a mixture according to distribution  $\mu$  (not necessarily a product distribution) is fed to both Alice's and Bob's input registers. He does not consider the case of superpositions being fed to input registers. For product distributions, our notion of privacy is more stringent than Klauck's, and in fact, the  $L^{\mathcal{P}}(f, \mu, A, B)$  defined above is an upper bound (to within an additive factor of  $\log |\mathcal{Z}|$ ) on Klauck's privacy loss function.
4. We restrict ourselves to product distributions because we allow Bob to cheat by putting a superposition in his input register  $Y$ . He should be able to do this without any *a priori* knowledge of  $x$ , which implies that the distribution  $\mu$  should be a product distribution. Also for defining privacy loss under non-product distributions on inputs, the initial mutual information between the correlated inputs of the two parties must be taken into account.
5. Consider a more general definition of privacy loss from Alice to Bob under a product distribution, where Alice is honest but Bob can cheat in any fashion so long as the reduced state of Alice's qubits after each round is the same as in the honest protocol. This definition is trivially an upper bound on the superpositional privacy loss, so tradeoffs proved in the setting of superpositional privacy loss continue to hold in the more general setting.

### 3.2 The privacy trade-off result

The theorem stated below is the formal version of Result 1 stated in the introduction.

**Theorem 1** *Consider a quantum protocol  $\mathcal{P}$  for  $\text{SetMemb}_n$  where Alice is given a subset of  $[n]$  and Bob an element of  $n$ . Let  $\mu$  denote the uniform probability distribution on Alice's and Bob's inputs. Suppose  $\mathcal{P}$  has error at most  $1/2 - \epsilon$  with respect to  $\mu$ . Suppose  $L^{\mathcal{P}}(\text{SetMemb}_n, \mu, B, A) \leq k$ . Then,*

$$L^{\mathcal{P}}(\text{SetMemb}_n, \mu, A, B) \geq \frac{n}{2^{\epsilon^{-3}(14k+25)}} - 2.$$

**Proof:** Let registers  $A, X, B, Y$  denote Alice's work qubits, Alice's input qubits, Bob's work qubits and Bob's input qubits respectively, at the end of protocol  $\mathcal{P}$ . We can assume without loss of generality that the last round of communication in  $\mathcal{P}$  is from Alice to Bob, since otherwise, we can add an extra round of communication at the end wherein Alice sends the answer qubit to Bob. This process increases  $L^{\mathcal{P}}(\text{SetMemb}_n, \mu, A, B)$  by at most two and does not increase  $L^{\mathcal{P}}(\text{SetMemb}_n, \mu, B, A)$  (see e.g. the information theoretic arguments in [CvDNT98]). Thus at the end of  $\mathcal{P}$ , Bob measures the answer qubit, which is a qubit in the register  $B$ , in the computational basis to determine  $f(x, y)$ .

Let  $|\psi_i\rangle_{XAYB}$  be the state vector of Alice's and Bob's qubits and  $(\rho_i)_{XA}$  the density matrix of Alice's qubits at the end of the protocol  $\mathcal{P}$ , when Alice is fed a uniform superposition over bit strings in her input register  $X$  and Bob is fed  $|i\rangle$  in his input register  $Y$ . In this proof, the subscripts of pure and mixed states denote the registers which are in those states. Let  $1/2 + \epsilon_i$  be the success probability of  $\mathcal{P}$  in this case. Without loss of generality,  $0 \leq \epsilon_i \leq 1/2$ . Consider a run, Run 1, of  $\mathcal{P}$  where a uniform mixture of indices is fed to register  $Y$ , and a uniform superposition over bit strings is fed to register  $X$ . Let  $1/2 + \epsilon$  be the success probability of  $\mathcal{P}$  for Run 1, which is also the success probability of  $\mathcal{P}$  with respect to  $\mu$ . Then  $\epsilon = (1/n) \sum_{i=1}^n \epsilon_i$ . Let  $I_1(Y : AX)$  denote the mutual information of register  $Y$  with registers  $AX$  at the end of Run 1 of  $\mathcal{P}$ , where the subscript 1 emphasises the fact that we are considering Run 1 of

$\mathcal{P}$ . We know that  $I_1(Y : AX) = L^{\mathcal{P}}(\text{SetMemb}_n, \mu, B, A) \leq k$ . Let  $\rho_{XA} := (1/n) \sum_{i=1}^n (\rho_i)_{XA}$  and  $k_i := S((\rho_i)_{XA} \| \rho_{XA})$ . Note that  $0 \leq k_i < \infty$  by Fact 4. By Fact 5,

$$k \geq I_1(Y : AX) = \frac{1}{n} \sum_{i=1}^n S((\rho_i)_{XA} \| \rho_{XA}) = \frac{1}{n} \sum_{i=1}^n k_i.$$

Let  $k'_i := k_i + 4\sqrt{k_i + 2} + 2\log(k_i + 2) + 6$  and  $r_i := (2/\epsilon_i)^2$ .

Let us now consider a run, Run 2, of  $\mathcal{P}$  with uniform superpositions fed to registers  $X, Y$ . Let  $|\phi\rangle_{XAYB}$  be the state vector of Alice's and Bob's qubits at the end of Run 2 of  $\mathcal{P}$ . Then,  $\text{Tr}_{YB} |\phi\rangle\langle\phi| = \rho_{XA}$ , and the success probability of  $\mathcal{P}$  for Run 2 is  $1/2 + \epsilon$ . Let  $Q$  be an additional qubit. By the substate theorem (Theorem 2, Sec. 4.2 which is a nuanced version of Result 2), there exist states  $|\psi'_i\rangle_{XAYBQ}, |\theta'_i\rangle_{XAYBQ}$  such that  $\| |\psi_i\rangle\langle\psi_i| - |\psi'_i\rangle\langle\psi'_i| \|_{\text{tr}} \leq 2/\sqrt{r_i} = \epsilon_i$  and  $\text{Tr}_{YBQ} |\phi_i\rangle\langle\phi_i| = \rho_{XA}$  where

$$|\phi_i\rangle_{XAYBQ} := \sqrt{\frac{r_i - 1}{r_i 2^{r_i k'_i}}} |\psi'_i\rangle_{XAYB} |1\rangle_Q + \sqrt{1 - \frac{r_i - 1}{r_i 2^{r_i k'_i}}} |\theta'_i\rangle_{XAYB} |0\rangle_Q,$$

In fact, there exists a unitary transformation  $U_i$  on registers  $YBQ$ , transforming the state  $|\phi\rangle_{XAYB} |0\rangle_Q$  to the state  $|\phi_i\rangle_{XAYBQ}$ .

For each  $i \in [n]$ , let  $X'_i$  denote the classical random variable got by measuring the  $i$ th bit of register  $X$  in state  $|\phi\rangle_{XAYB}$ . We now prove the following claim.

**Claim 1** For each  $i \in [n]$ , there is a POVM  $\mathcal{M}_i$  with three outcomes 0, 1, ? acting on  $YB$  such that if  $Z'_i$  is the result of  $\mathcal{M}_i$  on  $|\phi\rangle_{XAYB}$ , then  $\Pr[Z'_i \neq ?] \geq 2^{-4\epsilon_i^{-2}(k'_i+1)}$ , and  $\Pr[Z'_i = X'_i | Z'_i \neq ?] \geq 1/2 + \epsilon_i/2$ .

**Proof:** The POVM  $\mathcal{M}_i$  proceeds by first bringing in the ancilla qubit  $Q$  initialised to  $|0\rangle_Q$ , then applying  $U_i$  to the registers  $YBQ$  and finally measuring  $Q$  in the computational basis. If it observes  $|1\rangle_Q$ ,  $\mathcal{M}_i$  measures the answer qubit in  $B$  in the computational basis and declares the result as  $Z'_i$ . If it observes  $|0\rangle_Q$ ,  $\mathcal{M}_i$  outputs ?.

When applied to  $|\phi\rangle_{XAYB}$ ,  $\mathcal{M}_i$  first generates  $|\phi_i\rangle_{XAYBQ}$  and then measures  $Q$  in the computational basis. In the case when  $\mathcal{M}_i$  measures  $|1\rangle$  for qubit  $Q$ , which happens with probability

$$\Pr[Z'_i \neq ?] = \frac{r_i - 1}{r_i 2^{r_i k'_i}} \geq 2^{-4\epsilon_i^{-2}(k'_i+1)},$$

the state vector of  $XAYB$  collapses to  $|\psi'_i\rangle$ . In this case by Fact 1,

$$\Pr[Z'_i = X'_i | Z'_i \neq ?] \geq \frac{1}{2} + \epsilon_i - \frac{1}{2} \| |\psi_i\rangle\langle\psi_i| - |\psi'_i\rangle\langle\psi'_i| \|_{\text{tr}} \geq \frac{1}{2} + \frac{\epsilon_i}{2}.$$

■

Consider now a run, Run 3, of  $\mathcal{P}$  when a uniform mixture over bit strings is fed to register  $X$  and a uniform superposition over  $[n]$  is fed to register  $Y$ . Let  $\rho_{XAYB}$  denote the density matrix of the registers  $XAYB$  at the end of Run 3 of  $\mathcal{P}$ . In fact, measuring in the computational basis the register  $X$  in the state  $|\phi\rangle_{XAYB}$  gives us  $\rho_{XAYB}$ ; also,  $\text{Tr}_{YB} \rho_{XAYB} = \rho_{XA}$ . Let  $I_3(X : YB)$  denote the mutual information between register  $X$  and registers  $YB$  in the state  $\rho_{XAYB}$ , where the subscript 3 emphasises the fact that we are considering Run 3 of  $\mathcal{P}$ . For each  $i \in [n]$ , let  $X_i$  denote the classical random variable corresponding to the  $i$ th bit of register  $X$  in state  $\rho_{XAYB}$ . Then,  $X := X_1 \dots X_n$  is a uniformly distributed bit string of length  $n$ . Let  $Z_i$  denote the result of POVM  $\mathcal{M}_i$  of the above claim applied to  $\rho_{XAYB}$ . Then since  $\mathcal{M}_i$  acts

only on the registers  $YB$ , we get  $\Pr[Z_i \neq ?] = \Pr[Z'_i \neq ?] \geq 2^{-4\epsilon_i^{-2}(k'_i+1)}$ , and  $\Pr[Z_i = X_i \mid Z_i \neq ?] = \Pr[Z'_i = X'_i \mid Z_i \neq ?] \geq 1/2 + \epsilon_i/2$ . Define  $\text{Good} := \{i \in [n] : k_i \leq 2k/\epsilon, \epsilon_i \geq \epsilon/2\}$ . Since  $\forall i, k_i \geq 0$  and  $\epsilon_i \leq 1/2$ , by Markov's inequality,  $|\text{Good}| > n\epsilon/2$ . By Fact 6,

$$\begin{aligned} I_3(X : YB) &\geq \sum_{i=1}^n \frac{\epsilon_i^2 \cdot 2^{-4\epsilon_i^{-2}(k'_i+1)}}{4} \geq \sum_{i \in \text{Good}} \frac{\epsilon_i^2 \cdot 2^{-4\epsilon_i^{-2}(k'_i+1)}}{4} \\ &\geq \frac{n\epsilon^3 \cdot 2^{\epsilon^{-3}(2k+4\sqrt{2k+2}+2\log(2k+2)+7)}}{32} \geq \frac{n}{2^{\epsilon^{-3}(2k+4\sqrt{2k+2}+2\log(2k+2)+13)}} \\ &\geq \frac{n}{2^{\epsilon^{-3}(14k+25)}}. \end{aligned}$$

By the arguments in the first paragraph of this proof, we have  $L^{\mathcal{P}}(\text{SetMemb}_n, \mu, A, B) \geq I(X : YB) - 2$ . This completes the proof of the theorem.  $\blacksquare$

As we have mentioned earlier, this theorem has been generalised in [JRS05] in a suitable manner to relate the privacy loss for any relation in terms of its one-way communication complexity. We do not get into the details of this statement here. Instead, we give a weaker corollary of the present theorem that relates the privacy loss of a function to the *Vapnik-Chervonenkis dimension* (*VC-dimension*) of its communication matrix.

**Definition 9 (VC-dimension)** For a boolean valued function  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ , a set  $T \subseteq \mathcal{Y}$  is  $\mathcal{X}$ -shattered, if for all  $S \subseteq T$  there is an  $x \in \mathcal{X}$  such that  $\forall y \in T : f(x, y) = 1 \Leftrightarrow y \in S$ . The VC-dimension of  $f$  for  $\mathcal{X}$ ,  $\text{VC}_{\mathcal{X}}(f)$ , is the largest size of such a  $\mathcal{X}$ -shattered set  $T \subseteq \mathcal{Y}$ . Similarly, the VC-dimension of  $f$  for  $\mathcal{Y}$ ,  $\text{VC}_{\mathcal{Y}}(f)$ , is the largest size of a  $\mathcal{Y}$ -shattered set  $T \subseteq \mathcal{X}$ .

Informally,  $\text{VC}_{\mathcal{X}}(f)$  captures the size of the largest instance of the set membership problem  $\text{SetMemb}_n$  that can be ‘embedded’ into  $f$ . Using this connection, one can trivially prove a privacy trade-off result for  $f$  in terms of  $\text{VC}_{\mathcal{X}}(f)$ ,  $\text{VC}_{\mathcal{Y}}(f)$  by invoking Theorem 1. This generalises Klauck’s lower bound [Kla00] for the communication complexity of bounded error one-way quantum protocols for  $f$  in terms of its VC-dimension.

**Corollary 1** Let  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$  be a boolean valued function. Let  $\text{VC}_{\mathcal{X}}(f) = n$ . Then there is a product distribution  $\mu$  on  $\mathcal{X} \times \mathcal{Y}$  such that, if  $\mathcal{P}$  is a quantum protocol for  $f$  with average error at most  $1/2 - \epsilon$  with respect to  $\mu$ ,

$$L^{\mathcal{P}}(f, \mu, B, A) \leq k \Leftrightarrow L^{\mathcal{P}}(f, \mu, A, B) \geq \frac{n}{2^{\epsilon^{-3}(14k+25)}} - 2.$$

An analogous statement holds for  $\text{VC}_{\mathcal{Y}}(f)$ .

**Proof:** Since  $\text{VC}_{\mathcal{X}}(f) = n$ , there is a set  $T \subseteq \mathcal{Y}$ ,  $|T| = n$  which is shattered. Without loss of generality,  $T = [n]$ . For any subset  $S \subseteq T$ , there is an  $x \in \mathcal{X}$  such that  $\forall y' \in T : f(x, y') = 1 \Leftrightarrow y' \in S$ . We now give a reduction from  $\text{SetMemb}_n$  to  $f$  as follows: In  $\text{SetMemb}_n$ , Alice is given a subset  $S \subseteq [n]$  and Bob is given a  $y \in [n]$ . Alice and Bob run the protocol  $\mathcal{P}$  for  $f$  on inputs  $x$  and  $y$  respectively, to solve  $\text{SetMemb}_n$ . The corollary now follows from Theorem 1.  $\blacksquare$

The following consequence of Corr. 1 is immediate.

**Corollary 2** Quantum protocols for set membership  $\text{SetMemb}_n$ , set disjointness for subsets of  $[n]$  and inner product modulo 2 in  $\{0, 1\}^n$  each suffer from  $\Omega(\log n)$  privacy loss.

**Proof:** Follows trivially from Corr. 1 since all the three functions have VC-dimension  $n$ .  $\blacksquare$

## 4 The substate theorem

In this section, we prove the quantum substate theorem. But first, we state a fact from game theory that will be used in its proof.

### 4.1 A minimax theorem

We will require the following minimax theorem from game theory, which is a consequence of the Kakutani fixed point theorem in real analysis.

**Fact 7** *Let  $A_1, A_2$  be non-empty, convex and compact subsets of  $\mathbb{R}^n$  for some  $n$ . Let  $u : A_1 \times A_2 \rightarrow \mathbb{R}$  be a continuous function, such that*

- $\forall a_2 \in A_2$ , the set  $\{a_1 \in A_1 : \forall a'_1 \in A_1 u(a_1, a_2) \geq u(a'_1, a_2)\}$  is convex; i.e. for every  $a_2 \in A_2$ , the set of points  $a_1 \in A_1$ , such that  $u(a_1, a_2)$  is maximum is a convex set. And
- $\forall a_1 \in A_1$ , the set  $\{a_2 \in A_2 : \forall a'_2 \in A_2 u(a_1, a_2) \leq u(a_1, a'_2)\}$  is convex, i.e. for every  $a_1 \in A_1$ , the set of points  $a_2 \in A_2$ , such that  $u(a_1, a_2)$  is minimum is a convex set.

Then, there is an  $(a_1^*, a_2^*) \in A_1 \times A_2$  such that

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2) = u(a_1^*, a_2^*) = \min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2).$$

**Remark:** The above statement follows by combining Proposition 20.3 (which shows the existence of Nash equilibrium  $a^*$  in strategic games) and Proposition 22.2 (which connects Nash equilibrium and the min-max theorem for games defined using a pay-off function such as  $u$ ) of Osborne and Rubinstein's [OR94, pages 19–22] book on game theory.

### 4.2 Proof of the substate theorem

We now state the quantum substate theorem as it is actually used in our privacy lower bound proofs.

**Theorem 2 (Quantum substate theorem)** *Consider two Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ ,  $\dim(\mathcal{K}) \geq \dim(\mathcal{H})$ . Let  $\mathbb{C}^2$  denote the two dimensional complex Hilbert space. Let  $\rho, \sigma$  be density matrices in  $\mathcal{H}$ . Let  $r > 1$  be any real number. Let  $k := S(\rho||\sigma)$ . Let  $|\psi\rangle$  be a purification of  $\rho$  in  $\mathcal{H} \otimes \mathcal{K}$ . Then there exist pure states  $|\phi\rangle, |\theta\rangle \in \mathcal{H} \otimes \mathcal{K}$  and  $|\zeta\rangle \in \mathcal{H} \otimes \mathcal{K} \otimes \mathbb{C}^2$ , depending on  $r$ , such that  $|\zeta\rangle$  is a purification of  $\sigma$  and  $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\text{tr}} \leq 2/\sqrt{r}$ , where*

$$|\zeta\rangle := \sqrt{\frac{r-1}{r2^{rk'}}} |\phi\rangle|1\rangle + \sqrt{1 - \frac{r-1}{r2^{rk'}}} |\theta\rangle|0\rangle \quad \text{and} \quad k' := k + 4\sqrt{k+2} + 2\log(k+2) + 6.$$

**Remarks:**

1. Note that Result 2 in the introduction follows from above by tracing out  $\mathcal{K} \otimes \mathbb{C}^2$ .
2. From Result 2, one can easily see that  $\|\rho - \sigma\|_{\text{tr}} \leq 2 - 2^{-O(k)}$ . This implies a  $2^{-O(k)}$  lower bound on the fidelity of  $\rho$  and  $\sigma$ .

**Overview of the proof of Theorem 2:** As we have mentioned earlier, our proof of the quantum substate theorem goes through by first defining a new notion of distinguishability called *observational divergence*,  $D(\rho||\sigma)$ , between two density matrices  $\rho, \sigma$  in the same Hilbert space  $\mathcal{H}$ . Informally speaking, this notion

is a single observational version of relative entropy. Truly speaking, the substate theorem is a relationship between observational divergence and the substate condition. The main steps of our proof of the substate theorem are as follows:

1. We first prove an observational divergence 'lifting' theorem, Theorem 3, which shows that given two states  $\rho, \sigma$  in  $\mathcal{H}$  and any extension  $\sigma'$  of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$ ,  $\dim(\mathcal{K}) \geq \dim(\mathcal{H})$ , one can find a purification  $|\phi\rangle$  of  $\rho$  in  $\mathcal{H} \otimes \mathcal{K}$  such that  $D(|\phi\rangle\langle\phi||\sigma') = O(D(\rho||\sigma))$ . This result may be of independent interest.
2. Theorem 3 is in turn shown using Lemmas 1, 2. Lemma 1 can be thought of as an 'observational' substate result, i.e., a substate result with respect to a fixed POVM element, say  $F$ . We then prove Lemma 2 which, for any fixed  $0 \leq p \leq 1$ , removes the dependence on  $F$  if  $F$  satisfies  $\text{Tr}(F|\psi\rangle\langle\psi|) \geq p$ . The dependence on parameter  $p$  is finally removed by performing a 'discrete integration' operation, giving us Theorem 3.
3. Theorem 3 helps us reduce the final statement we intend to prove (Theorem 2) to the case when  $\rho$  is a pure state. The substate theorem when  $\rho$  is a pure state is shown in Lemma 3. In the proof of Lemma 3, we further reduce this case to analysing only a two dimensional scenario which is then resolved by a direct calculation.
4. Finally Theorem 2, that is, the quantum substate theorem for relative entropy is established by using Proposition 2 which shows that observational divergence is never much bigger than relative entropy for any pair of states.

Let us begin by defining observational divergence.

**Definition 10 (Observational divergence)** *Let  $\rho, \sigma$  be density matrices in the same Hilbert space  $\mathcal{H}$ . Their observational divergence is defined as*

$$D(\rho||\sigma) := \sup_F \left( \text{Tr}(F\rho) \log \frac{\text{Tr}(F\rho)}{\text{Tr}(F\sigma)} \right),$$

where  $F$  above ranges over POVM elements on  $\mathcal{H}$  such that  $\text{Tr}(F\sigma) \neq 0$ .

The following properties of observational divergence follow easily from the definition.

**Proposition 1** *Let  $\rho, \sigma$  be density matrices in the same Hilbert space  $\mathcal{H}$ . Then*

1.  $D(\rho||\sigma) \geq 0$ , with equality iff  $\rho = \sigma$ .
2.  $D(\rho||\sigma) < +\infty$  iff  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ . If  $D(\rho||\sigma) < +\infty$ , then there is a POVM element  $F$  which achieves equality in Definition 10.
3.  $D(\cdot||\cdot)$  is continuous in its two arguments over the domain of pairs of states  $(\rho, \sigma)$  such that  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ ;
4. (Unitary invariance) If  $U$  is a unitary transformation on  $\mathcal{H}$ ,  $D(U\rho U^\dagger||U\sigma U^\dagger) = D(\rho||\sigma)$ .
5. (Monotonicity) Suppose  $\mathcal{K}$  is a Hilbert space, and  $\rho', \sigma'$  are extensions of  $\rho, \sigma$  in  $\mathcal{H} \otimes \mathcal{K}$ . Then,  $D(\rho'||\sigma') \geq D(\rho||\sigma)$ . This implies, via unitary invariance and the Kraus representation theorem, that if  $\mathcal{T}$  is a completely positive trace preserving superoperator from  $\mathcal{H}$  to a Hilbert space  $\mathcal{L}$ , then  $D(\mathcal{T}\rho||\mathcal{T}\sigma) \leq D(\rho||\sigma)$ .

Fact 4 and Proposition 1 seem to suggest that relative entropy and observational divergence are similar quantities. In fact, the relative entropy is an upper bound on the observational divergence to within an additive constant. More properties of observational divergence as well as comparisons with relative entropy are discussed in the appendix.

**Proposition 2** *Let  $\rho, \sigma$  be density matrices in the same Hilbert space  $\mathcal{H}$ . Then,  $D(\rho||\sigma) < S(\rho||\sigma) + 1$ .*

**Proof:** By Fact 4 and Proposition 1,  $D(\rho||\sigma) = +\infty$  iff  $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$  iff  $S(\rho||\sigma) = +\infty$ . Thus, we can henceforth assume without loss of generality that  $D(\rho||\sigma) < +\infty$ . By Proposition 1, there is a POVM element  $F$  such that  $D(\rho||\sigma) = p \log(p/q)$ , where  $p := \text{Tr}(F\rho)$  and  $q := \text{Tr}(F\sigma)$ . We now have

$$\begin{aligned} S(\rho||\sigma) &\geq p \log \frac{p}{q} + (1-p) \log \frac{(1-p)}{(1-q)} > p \log \frac{p}{q} + (1-p) \log \frac{1}{(1-q)} - 1 \geq p \log \frac{p}{q} - 1 \\ &= D(\rho||\sigma) - 1. \end{aligned}$$

The first inequality follows from the Lindblad-Uhlmann monotonicity of relative entropy (Fact 4), and the second inequality follows because  $(1-p) \log(1-p) \geq (-\log e)/e > -1$ , for  $0 \leq p \leq 1$ . This completes the proof of the lemma.  $\blacksquare$

We next prove the ‘observational’ substate lemma for a fixed POVM element, as mentioned before in the overview.

**Lemma 1** *Consider two Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ ,  $\dim(\mathcal{K}) \geq \dim(\mathcal{H})$ . Let  $\rho, \sigma$  be density matrices in  $\mathcal{H}$ . Let  $|\psi\rangle$  be a purification of  $\rho$  in  $\mathcal{H} \otimes \mathcal{K}$ . Let  $F$  be a POVM element on  $\mathcal{H} \otimes \mathcal{K}$ . Let  $\beta > 1$ . Then there exists a purification  $|\phi\rangle$  of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$  such that  $q \geq \frac{p}{2^{k'/p}}$ , where  $p := \text{Tr}(F|\psi\rangle\langle\psi|)$ ,  $q := \text{Tr}(F|\phi\rangle\langle\phi|)$  and  $k' := \beta D(\rho||\sigma) - 2 \log(1 - \beta^{-1/2})$ .*

**Proof:** We assume without loss of generality that  $0 < D(\rho||\sigma) < +\infty$  and that  $p > 0$ . Let  $n := \dim(\mathcal{H} \otimes \mathcal{K})$  and  $\{|\alpha_i\rangle\}_{i=1}^n$  be the orthonormal eigenvectors of  $F$  with corresponding eigenvalues  $\{\lambda_i\}_{i=1}^n$ . Note that  $0 \leq \lambda_i \leq 1$  and  $|\alpha_i\rangle \in \mathcal{H} \otimes \mathcal{K}$ . We have from definition of  $p$ ,

$$p = \sum_{i=1}^n \lambda_i |\langle\alpha_i|\psi\rangle|^2. \quad (4)$$

Define,

$$|\theta'\rangle := \frac{\sum_{i=1}^n \lambda_i \langle\alpha_i|\psi\rangle |\alpha_i\rangle}{\sqrt{p}} \quad \text{and} \quad |\theta\rangle := \frac{|\theta'\rangle}{\| |\theta'\rangle \|}. \quad (5)$$

Note that  $p = |\langle\psi|\theta\rangle|^2 \| |\theta'\rangle \|^2$  and  $0 < \| |\theta'\rangle \|^2 \leq 1$ . We will show later that there exists a purification  $|\phi\rangle$  of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$  such that

$$|\langle\phi|\theta\rangle|^2 \geq \frac{|\langle\psi|\theta\rangle|^2}{2^{k'/|\langle\psi|\theta\rangle|^2}}. \quad (6)$$

This would imply,

$$\begin{aligned}
\frac{p}{2^{k'/p}} &= \frac{|\langle \psi | \theta \rangle|^2 \|\theta'\|^2}{2^{k'/(\|\langle \psi | \theta \rangle\|^2 \|\theta'\|^2)}} \leq \frac{|\langle \psi | \theta \rangle|^2 \|\theta'\|^2}{2^{k'/\|\langle \psi | \theta \rangle\|^2}} && \text{(since } 0 < \|\theta'\|^2 \leq 1) \\
&\leq |\langle \phi | \theta \rangle|^2 \|\theta'\|^2 && \text{(from Eq. (6))} \\
&= |\langle \phi | \theta' \rangle|^2 = \frac{|\sum_{i=1}^n \lambda_i \langle \alpha_i | \psi \rangle \langle \phi | \alpha_i \rangle|^2}{p} && \text{(from Eq. (5))} \\
&= \frac{|\sum_{i=1}^n \lambda_i \langle \alpha_i | \psi \rangle \langle \phi | \alpha_i \rangle|^2}{\sum_{i=1}^n \lambda_i |\langle \alpha_i | \psi \rangle|^2} && \text{(from Eq. (4))} \\
&\leq \sum_{i=1}^n \lambda_i |\langle \alpha_i | \phi \rangle|^2 && \text{(from the Cauchy-Schwarz inequality)} \\
&= \text{Tr}(F|\phi\rangle\langle\phi|),
\end{aligned}$$

and we would be done.

For getting the desired  $|\phi\rangle$  as Eq. (6), let us first define the density matrix  $\tau$  in  $\mathcal{H}$  as  $\tau := \text{Tr}_{\mathcal{K}} |\theta\rangle\langle\theta|$ . By Facts 2 and 3, there is a purification  $|\phi\rangle$  of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$  and a POVM  $\{F_1, \dots, F_l\}$  in  $\mathcal{H}$  such that,

$$|\langle \phi | \theta \rangle| = B(\tau, \sigma) = \sum_{i=1}^l \sqrt{c_i b_i},$$

where  $c_i := \text{Tr}(F_i \tau)$  and  $b_i := \text{Tr}(F_i \sigma)$ . Let  $a_i := \text{Tr}(F_i \rho)$ . We know from Facts 2 and 3 that

$$0 < \sqrt{p} \leq |\langle \psi | \theta \rangle| \leq B(\tau, \rho) \leq \sum_{i=1}^l \sqrt{c_i a_i}.$$

Note that the  $a_i$ 's are non-negative real numbers summing up to 1, and so are the  $b_i$ 's and the  $c_i$ 's.

For  $\beta > 1$ , define the set  $S_\beta := \{i \in [l] : a_i > b_i \cdot 2^{\beta k / B(\tau, \rho)^2}\}$ , where  $k := D(\rho \| \sigma)$ . Note that  $\forall i \in S, b_i \neq 0$  as  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ ,  $k$  being finite. Define the POVM element  $G$  on  $\mathcal{H}$  as  $G := \sum_{i \in S_\beta} F_i$ . Let  $a := \text{Tr}(G\rho)$  and  $b := \text{Tr}(G\sigma)$ . Then  $a = \sum_{i \in S_\beta} a_i$ ,  $b = \sum_{i \in S_\beta} b_i$ ,  $b > 0$  and  $a > b \cdot 2^{\beta k / B(\tau, \rho)^2}$ . We have that

$$D(\rho \| \sigma) = k \geq a \log \frac{a}{b} > \frac{\beta k a}{B(\tau, \rho)^2} \Rightarrow a < \frac{B(\tau, \rho)^2}{\beta}.$$

Now, by the Cauchy-Schwarz inequality and the other inequalities proved above, we get

$$\begin{aligned}
B(\tau, \rho) &\leq \sum_{i=1}^l \sqrt{c_i a_i} = \sum_{i \in S_\beta} \sqrt{c_i a_i} + \sum_{i \notin S_\beta} \sqrt{c_i a_i} \\
&\leq \sqrt{\sum_{i \in S_\beta} c_i} \sqrt{\sum_{i \in S_\beta} a_i} + 2^{\beta k / (2B(\tau, \rho)^2)} \sum_{i \notin S_\beta} \sqrt{c_i b_i} \leq 1 \cdot \sqrt{a} + 2^{\beta k / (2B(\tau, \rho)^2)} B(\tau, \sigma) \\
&< \frac{B(\tau, \rho)}{\sqrt{\beta}} + 2^{\beta k / (2B(\tau, \rho)^2)} B(\tau, \sigma).
\end{aligned}$$

This shows that

$$B(\tau, \rho)^2 < (1 - \beta^{-1/2})^{-2} \cdot 2^{\beta k / B(\tau, \rho)^2} B(\tau, \sigma)^2 \Rightarrow |\langle \psi | \theta \rangle|^2 < (1 - \beta^{-1/2})^{-2} \cdot 2^{\beta k / \|\langle \psi | \theta \rangle\|^2} |\langle \phi | \theta \rangle|^2.$$

Since  $k' = \beta k - 2 \log(1 - \beta^{-1/2})$ , we get  $|\langle \phi | \theta \rangle|^2 \geq \frac{|\langle \psi | \theta \rangle|^2}{2^{k'/1} |\langle \psi | \theta \rangle|^2}$ , completing the proof of the lemma.  $\blacksquare$

In the previous lemma, the purification  $|\phi\rangle$  of  $\sigma$  was a function of the POVM element  $F$ . We now prove a lemma which, for any fixed  $0 \leq p \leq 1$ , removes the dependence on  $F$  satisfying  $\text{Tr}(F|\psi\rangle\langle\psi|) \geq p$ , at the expense of having a, in general, mixed extension of  $\sigma$  in the place of a pure extension i.e. purification.

**Lemma 2** Consider two Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ ,  $\dim(\mathcal{K}) \geq \dim(\mathcal{H})$ . Let  $\rho, \sigma$  be density matrices in  $\mathcal{H}$  and  $|\psi\rangle$  be a purification of  $\rho$  in  $\mathcal{H} \otimes \mathcal{K}$ . Let  $0 \leq p \leq 1$  and  $\beta > 1$ . Then there exists an extension  $\omega$  of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$  such that for all POVM elements  $F$  on  $\mathcal{H} \otimes \mathcal{K}$  such that  $\text{Tr}(F|\psi\rangle\langle\psi|) \geq p$ ,  $\text{Tr}(F\omega) \geq p/2^{k'/p}$ , where  $k' := \beta D(\rho\|\sigma) - 2 \log(1 - \beta^{-1/2})$ .

**Proof:** We assume without loss of generality that  $0 < D(\rho\|\sigma) < +\infty$  and that  $p > 0$ . Consider the set  $A_1$  of all extensions  $\omega$  of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$  and the set  $A_2$  of all POVM operators  $F$  in  $\mathcal{H} \otimes \mathcal{K}$  such that  $\text{Tr}(F|\psi\rangle\langle\psi|) \geq p$ . Observe that  $A_1, A_2$  are non-empty, compact, convex sets. Without loss of generality,  $A_2$  is non-empty. Thus from Lemma 1, for every  $F \in A_2$ , we have a purification  $|\phi^F\rangle \in \mathcal{H} \otimes \mathcal{K}$  of  $\sigma$  such that

$$\text{Tr}(F|\phi^F\rangle\langle\phi^F|) \geq \frac{\text{Tr}(F|\psi\rangle\langle\psi|)}{2^{k'/\text{Tr}(F|\psi\rangle\langle\psi|)}} \geq \frac{p}{2^{k'/p}}.$$

Let us define  $u : A_1 \times A_2 \rightarrow \mathbb{R}$  as  $\forall \omega \in A_1, \forall F \in A_2, u(\omega, F) = \text{Tr} F\omega$ . Note that with these definitions, the conditions of Fact 7 are trivially satisfied (note that we think of our matrices, which in general have complex entries, as vectors in a larger real vector space). Therefore, using Fact 7, we see that there exists an extension  $\omega$  of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$  such that  $\text{Tr}(F\omega) \geq \frac{p}{2^{k'/p}}$  for all  $F \in A_2$ . This completes the proof.  $\blacksquare$

The previous lemma depends upon the parameter  $p$ . We now remove this restriction by performing a ‘discrete integration’ operation and obtain the ‘observational divergence lifting’ result. This result relates the observational divergence of a pair of density matrices to the observational divergence of their extensions in an extended Hilbert space, where the extension of the first density matrix is a pure state. This may be of independent interest.

**Theorem 3 (Observational divergence lifting)** Consider two Hilbert spaces  $\mathcal{H}, \mathcal{K}$ ,  $\dim(\mathcal{K}) \geq \dim(\mathcal{H})$ . Let  $\rho, \sigma$  be density matrices in  $\mathcal{H}$ , and  $|\psi\rangle$  be a purification of  $\rho$  in  $\mathcal{H} \otimes \mathcal{K}$ . Then there exists an extension  $\omega$  of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$  such that  $D(|\psi\rangle\langle\psi| \|\omega) < D(\rho\|\sigma) + 4\sqrt{D(\rho\|\sigma) + 1} + 2 \log(D(\rho\|\sigma) + 1) + 5$ .

**Proof:** We assume without loss of generality that  $0 < D(\rho\|\sigma) < +\infty$ . Let  $\beta > 1$  and  $\gamma \geq 1$ . Define the monotonically increasing function  $f : [0, 1] \rightarrow [0, 1]$  as follows:

$$f(p) := \frac{p}{2^{k'/p}} \quad \text{where } 0 \leq p \leq 1 \quad \text{and} \quad k' := \beta D(\rho\|\sigma) - 2 \log(1 - \beta^{-1/2}).$$

For a fixed positive integer  $l$ , define  $T_\gamma(l) := \sum_{i=1}^l l^{\gamma-1}$ . It is easy to see by elementary calculus that  $\gamma^{-1} \cdot l^\gamma \leq T_\gamma(l) \leq \gamma^{-1} \cdot (l+1)^\gamma$ . Define the density matrix  $\omega_l$  in  $\mathcal{H} \otimes \mathcal{K}$  as  $\omega_l := (T_\gamma(l))^{-1} \sum_{i=1}^l i^{\gamma-1} \omega(i/l)$ , where for  $0 \leq p \leq 1$ ,  $\omega(p)$  is an extension of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$  such that  $\text{Tr}(F\omega(p)) \geq f(p)$  for all POVM elements  $F$  on  $\mathcal{H} \otimes \mathcal{K}$  satisfying  $\text{Tr}(F|\psi\rangle\langle\psi|) \geq p$ . Such an  $\omega(p)$  exists by Lemma 2. Then,  $\text{Tr}_{\mathcal{K}} \omega_l = \sigma$  i.e.  $\omega_l$  is an extension of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K}$ .

Suppose  $F$  is a POVM element on  $\mathcal{H} \otimes \mathcal{K}$ . Define  $p := \text{Tr}(F|\psi\rangle\langle\psi|)$ . Choose an integer  $l$  satisfying  $l \geq 1/p$ . Let  $j/l \leq p < (j+1)/l$ , where  $j$  is an integer satisfying  $1 \leq j \leq l$ . We assume without loss of

generality that  $p > 0$ . Then,

$$\begin{aligned}
\mathrm{Tr}(F\omega_l) &= \frac{1}{T_\gamma(l)} \sum_{i=1}^j i^{\gamma-1} \cdot \mathrm{Tr}(F\omega(i/l)) \\
&\geq \frac{1}{T_\gamma(l)} \sum_{i=1}^j i^{\gamma-1} \cdot f(i/l) \\
&\geq \frac{T_\gamma(j)}{T_\gamma(l)} \cdot f\left(\frac{1}{T_\gamma(j)} \sum_{i=1}^j \frac{i^\gamma}{l}\right) = \frac{T_\gamma(j)}{T_\gamma(l)} \cdot f\left(\frac{T_{\gamma+1}(j)}{l \cdot T_\gamma(j)}\right) \\
&\geq \left(\frac{j}{l+1}\right)^\gamma \cdot f\left(\frac{\gamma \cdot j^{\gamma+1}}{l(\gamma+1) \cdot (j+1)^\gamma}\right) \\
&\geq \left(\frac{pl-1}{l+1}\right)^\gamma \cdot f\left(\left(\frac{\gamma(pl-1)}{(\gamma+1)l}\right) \left(\frac{pl-1}{pl+1}\right)^\gamma\right).
\end{aligned}$$

The second inequality above follows from the convexity of  $f(\cdot)$ . By compactness, the set  $\{\omega_l : l \in \mathbb{N}\}$  has limit points. Choose a limit point  $\omega$ . By standard continuity arguments,  $\mathrm{Tr}_K \omega = \sigma$  and

$$\begin{aligned}
q &:= \mathrm{Tr}(F\omega) \geq \lim_{l \rightarrow +\infty} \left[ \left(\frac{pl-1}{l+1}\right)^\gamma \cdot f\left(\left(\frac{\gamma(pl-1)}{(\gamma+1)l}\right) \left(\frac{pl-1}{pl+1}\right)^\gamma\right) \right] = p^\gamma \cdot f\left(\frac{\gamma p}{\gamma+1}\right) \\
&= \frac{\gamma \cdot p^{\gamma+1}}{(\gamma+1) \cdot 2^{k'(\gamma+1)\gamma^{-1}p^{-1}}}.
\end{aligned}$$

Hence,  $q > 0$  and

$$\begin{aligned}
p \log \frac{p}{q} &\leq p \log \left( \gamma^{-1}(\gamma+1) \cdot p^{-\gamma} \cdot 2^{k'(\gamma+1)\gamma^{-1}p^{-1}} \right) = p \log(1 + \gamma^{-1}) - \gamma p \log p + (1 + \gamma^{-1})k' \\
&< (1 + \gamma^{-1})k' + \gamma + 1.
\end{aligned}$$

The second inequality follows because  $-p \log p < 1$  for  $0 \leq p \leq 1$ , and  $\log(1 + \gamma^{-1}) \leq 1$  for all  $\gamma \geq 1$ . Substituting  $k' = \beta D(\rho\|\sigma) - 2 \log(1 - \beta^{-1/2})$  gives

$$D(|\psi\rangle\langle\psi| \|\omega) < \beta(1 + \gamma^{-1})D(\rho\|\sigma) - 2(1 + \gamma^{-1}) \log(1 - \beta^{-1/2}) + \gamma + 1.$$

We set  $\beta = (1 + (D(\rho\|\sigma) + 1)^{-1/2})^2$  and  $\gamma = (D(\rho\|\sigma) + 1)^{1/2}$  to get

$$\begin{aligned}
D(|\psi\rangle\langle\psi| \|\omega) &< (1 + (D(\rho\|\sigma) + 1)^{-1/2})^3 \cdot D(\rho\|\sigma) + (D(\rho\|\sigma) + 1)^{1/2} + 1 \\
&\quad - 2(1 + (D(\rho\|\sigma) + 1)^{-1/2}) \cdot \log \frac{(D(\rho\|\sigma) + 1)^{-1/2}}{1 + (D(\rho\|\sigma) + 1)^{-1/2}} \\
&< (1 + (D(\rho\|\sigma) + 1)^{-1/2})^3 \cdot D(\rho\|\sigma) + (D(\rho\|\sigma) + 1)^{1/2} + 1 \\
&\quad + (1 + (D(\rho\|\sigma) + 1)^{-1/2}) \cdot \log(D(\rho\|\sigma) + 1) \\
&< D(\rho\|\sigma) + 4\sqrt{D(\rho\|\sigma) + 1} + (1 + (D(\rho\|\sigma) + 1)^{-1/2}) \cdot \log(D(\rho\|\sigma) + 1) + 5 \\
&< D(\rho\|\sigma) + 4\sqrt{D(\rho\|\sigma) + 1} + 2 \log(D(\rho\|\sigma) + 1) + 5.
\end{aligned}$$

This completes the proof of the lemma. ■

We now prove the following lemma, which can be thought of as a substate result when the first density matrix is in fact a pure state.

**Lemma 3** Let  $|\psi\rangle$  be a pure state and  $\sigma$  be a density matrix in the same Hilbert space  $\mathcal{H}$ . Let  $k := D(|\psi\rangle\langle\psi||\sigma)$ . Then for all  $r \geq 1$ , there exists a pure state  $|\phi\rangle$ , depending on  $r$ , such that

$$\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\text{tr}} < \frac{2}{\sqrt{r}} \quad \text{and} \quad \left( \frac{r-1}{r2^{rk}} \right) |\phi\rangle\langle\phi| < \sigma.$$

**Proof:** We assume without loss of generality that  $0 < k < +\infty$ . Consider  $M := \sigma - (|\psi\rangle\langle\psi|/2^{rk})$ . We claim that  $M$  has at most one negative eigenvalue. This is because, from Weyl's inequality [Bha97, Theorem III.2.1], no eigenvalue of a matrix decreases on adding a positive semi-definite matrix. Since  $\sigma$  is positive semi-definite, adding it to  $-(|\psi\rangle\langle\psi|/2^{rk})$  to produce  $M$ , cannot decrease any eigenvalue. Hence at most one eigenvalue of  $M$  can be negative, since  $-(|\psi\rangle\langle\psi|/2^{rk})$  has exactly one negative eigenvalue viz.  $-1/2^{rk}$ , and the rest are zero.

If  $M \geq 0$  we take  $|\phi\rangle$  to be  $|\psi\rangle$ . The lemma trivially holds in this case.

Otherwise, let  $|w\rangle$  be the eigenvector corresponding to the unique negative eigenvalue  $-\alpha$  of  $M$ . Thinking of  $|w\rangle\langle w|$  as a POVM element, we get

$$0 > -\alpha = \text{Tr}(M|w\rangle\langle w|) = \langle w|\sigma|w\rangle - \frac{|\langle\psi|w\rangle|^2}{2^{rk}} \Rightarrow \langle w|\sigma|w\rangle < \frac{|\langle\psi|w\rangle|^2}{2^{rk}}.$$

Hence

$$k = D(|\psi\rangle\langle\psi||\sigma) \geq |\langle\psi|w\rangle|^2 \log \frac{|\langle\psi|w\rangle|^2}{\langle w|\sigma|w\rangle} > rk|\langle\psi|w\rangle|^2 \Rightarrow |\langle\psi|w\rangle|^2 < \frac{1}{r} \leq 1.$$

In particular, this shows that  $|\psi\rangle, |w\rangle$  are linearly independent.

Let  $n := \dim(\mathcal{H})$ . Let  $\{|v\rangle, |w\rangle\}$  be an orthonormal basis for the two dimensional subspace of  $\mathcal{H}$  spanned by  $\{|\psi\rangle, |w\rangle\}$ . Extend it to  $\{|v_1\rangle, \dots, |v_{n-2}\rangle, |v\rangle, |w\rangle\}$ , an orthonormal basis for the entire space  $\mathcal{H}$ . In this basis we have the following matrix equation,

$$\begin{bmatrix} F & e & d \\ e^\dagger & a & b \\ d^\dagger & b^\dagger & c \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0^\dagger & x & y \\ 0^\dagger & y^\dagger & z \end{bmatrix} = \begin{bmatrix} P & l \\ l^\dagger & -\alpha \end{bmatrix}, \quad (7)$$

where the first, second and third matrices are  $\sigma$ ,  $|\psi\rangle\langle\psi|/2^{rk}$  and  $M$  respectively.  $F$  is an  $(n-2) \times (n-2)$  matrix,  $P$  is an  $(n-1) \times (n-1)$  matrix,  $d, e$  are  $(n-2) \times 1$  matrices and  $l$  is an  $(n-1) \times 1$  matrix.  $a, c, x, z, \alpha$  are non-negative real numbers and  $b, y$  are complex numbers. The zeroes above denote all zero matrices of appropriate dimensions. The dagger denotes conjugate transpose.

**Claim 2** We have the following properties.

1.  $b, y \in \mathbb{C}$ ,  $a, c, x, z, \alpha \in \mathbb{R}$ .
2.  $b = y \neq 0$ ,  $1/(r2^{rk}) > z = c + \alpha > c > 0$ ,  $\alpha > 0$ ,  $a > 0$ ,  $0 < x < 1/2^{rk}$ ,  $x + z = 1/2^{rk}$ ,  $l = 0$  and  $d = 0$ .
3.  $0 < \frac{xc}{|b|^2} < \frac{xz}{|y|^2} = 1$ .

**Proof:** The first part of the claim has already been mentioned above. Since  $|w\rangle$  is an eigenvector of  $M$  corresponding to eigenvalue  $-\alpha$ ,  $l = 0$ . By inspection, we have  $b = y, z = c + \alpha, d = 0$ . We have  $x > 0$  since  $|\psi\rangle, |w\rangle$  are linearly independent, and  $z > c \geq 0$  since  $\alpha > 0$ . Now,  $x + z = \text{Tr}(|\psi\rangle\langle\psi|/2^{rk}) = 1/2^{rk}$  and so  $x < 1/2^{rk}$ . Also,  $z = |\langle\psi|w\rangle|^2/2^{rk} < 1/(r2^{rk})$ . Since  $\sigma \geq 0, F \geq 0$  and  $\begin{bmatrix} a & b \\ b^\dagger & c \end{bmatrix} \geq 0$ . Hence,

$$\det \begin{bmatrix} a & b \\ b^\dagger & c \end{bmatrix} = ac - |b|^2 \geq 0.$$

Since  $|\psi\rangle\langle\psi|/2^{rk}$  has one dimensional support,

$$\det \begin{bmatrix} x & y \\ y^\dagger & z \end{bmatrix} = xz - |y|^2 = 0.$$

If  $c = 0$  then  $y = b = 0$ , which implies that  $xz = 0$ , which is a contradiction. Hence,  $c > 0$  and  $b \neq 0$ . Similarly,  $a > 0$ . This proves the second part of the claim. The third part now follows easily.  $\blacksquare$

We can now write  $\sigma = \sigma_1 + \sigma_2$ , where

$$\sigma_1 := \begin{bmatrix} F & e & 0 \\ e^\dagger & a - \frac{|b|^2}{c} & 0 \\ 0^\dagger & 0^\dagger & 0 \end{bmatrix} \quad \text{and} \quad \sigma_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0^\dagger & \frac{|b|^2}{c} & b \\ 0^\dagger & b^\dagger & c \end{bmatrix}.$$

Note that  $|\xi\rangle = (0, \dots, 0, 1, -b^\dagger/c)$  is an eigenvector of  $\sigma_2$  corresponding to the eigenvalue 0. We have  $\sigma_2 \geq 0$ , and in fact,  $\sigma_2$  has one dimensional support. We now claim that  $\sigma_1 \geq 0$ . For otherwise, since  $F \geq 0$ , there is a vector  $|\theta\rangle$  of the form  $(a_1, \dots, a_{n-2}, 1, 0)$  such that  $\langle\theta|\sigma_1|\theta\rangle < 0$ . Now consider the vector  $|\theta'\rangle := (a_1, \dots, a_{n-2}, 1, -b^\dagger/c)$ . We have,

$$\langle\theta'|\sigma|\theta'\rangle = \langle\theta'|\sigma_1|\theta'\rangle + \langle\theta'|\sigma_2|\theta'\rangle = \langle\theta|\sigma_1|\theta\rangle + \langle\xi|\sigma_2|\xi\rangle < 0,$$

contradicting  $\sigma \geq 0$ . This shows that  $\sigma_1 \geq 0$ , and hence,  $\sigma \geq \sigma_2$ .

We are now finally in a position to define the pure state  $|\phi\rangle$ . Note that  $|\phi\rangle\langle\phi|$  is nothing but  $\sigma_2$  normalised to have unit trace. That is,

$$|\phi\rangle\langle\phi| := \frac{\sigma_2}{\frac{|b|^2}{c} + c}.$$

Using Claim 2 we get,

$$\text{Tr} \sigma_2 = \frac{|b|^2}{c} + c > \frac{|b|^2}{z} + c = x + z - \alpha > \frac{r-1}{r2^{rk}}.$$

Hence,  $\frac{r-1}{r2^{rk}}|\phi\rangle\langle\phi| < \sigma_2 \leq \sigma$ . This shows the second assertion of the lemma.

To complete the proof of the lemma, we still need to show that  $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\text{tr}}$  is small. Up to global phase factors, one can write  $|\psi\rangle, |\phi\rangle$  as follows:

$$|\psi\rangle = \frac{\frac{b}{\sqrt{z}}|v\rangle + \sqrt{z}|w\rangle}{\sqrt{\frac{|b|^2}{z} + z}}, \quad |\phi\rangle = \frac{\frac{b}{\sqrt{c}}|v\rangle + \sqrt{c}|w\rangle}{\sqrt{\frac{|b|^2}{c} + c}}.$$

We now lower bound  $|\langle\phi|\psi\rangle|$  as follows, using Claim 2.

$$\begin{aligned} |\langle\phi|\psi\rangle| &= \frac{\frac{|b|^2}{\sqrt{cz}} + \sqrt{cz}}{\sqrt{\frac{|b|^2}{c} + c \cdot \sqrt{\frac{|b|^2}{z} + z}}} = \frac{|b|^2 + cz}{\sqrt{(|b|^2 + c^2)(|b|^2 + z^2)}} \\ &> \frac{|b|^2 + cz}{\sqrt{(|b|^2 + cz)(|b|^2 + z^2)}} = \sqrt{\frac{|b|^2 + cz}{|b|^2 + z^2}} = \sqrt{\frac{x + c}{x + z}} = \sqrt{1 - \frac{\alpha}{x + z}} \\ &> \sqrt{1 - \frac{1}{r}}. \end{aligned}$$

This proves that  $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\text{tr}} = 2\sqrt{1 - |\langle\phi|\psi\rangle|^2} < 2/\sqrt{r}$ , establishing the first assertion of the lemma and completing its proof.  $\blacksquare$

Using Theorem 3, Lemma 3 and Proposition 2, we are now finally in a position to prove the quantum substate theorem.

**Proof (Theorem 2):** By Proposition 2 and Theorem 3, there exists a density matrix  $\omega$  in  $\mathcal{H} \otimes \mathcal{K}$  such that  $\text{Tr}_{\mathcal{K}} \omega = \sigma$  and

$$\begin{aligned} D(|\psi\rangle\langle\psi| \| \omega) &< D(\rho \| \sigma) + 4\sqrt{D(\rho \| \sigma) + 1} + 2\log(D(\rho \| \sigma) + 1) + 5 \\ &< S(\rho \| \sigma) + 4\sqrt{S(\rho \| \sigma) + 2} + 2\log(S(\rho \| \sigma) + 2) + 6 = k'. \end{aligned}$$

By Lemma 3, there exists a pure state  $|\phi\rangle$  such that

$$\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\text{tr}} \leq \frac{2}{\sqrt{r}} \quad \text{and} \quad \left(\frac{r-1}{r2^{rk'}}\right) |\phi\rangle\langle\phi| \leq \omega.$$

Let  $\tau_1 := \text{Tr}_{\mathcal{K}} |\phi\rangle\langle\phi|$ . By above,  $\left(\frac{r-1}{r2^{rk'}}\right) \tau_1 \leq \sigma$ . That is, there exists a density matrix  $\tau_2$  in  $\mathcal{H}$  such that

$$\sigma = \left(\frac{r-1}{r2^{rk'}}\right) \tau_1 + \left(1 - \frac{r-1}{r2^{rk'}}\right) \tau_2.$$

Let  $|\theta\rangle \in \mathcal{H} \otimes \mathcal{K}$  be a canonical purification of  $\tau_2$ . Then,  $|\zeta\rangle$  defined in the statement of Theorem 2 is a purification of  $\sigma$  in  $\mathcal{H} \otimes \mathcal{K} \otimes \mathbb{C}^2$ . This completes the proof of Theorem 2.  $\blacksquare$

## 5 Conclusion and open problems

In this paper we have proved a theorem about relative entropy of quantum states which gives a novel interpretation to this information theoretic quantity. Using this theorem, we have shown a privacy trade-off for computing set membership in the two-party quantum communication model.

The statements of the classical and quantum substate theorems have one important difference. For two quantum states  $\rho, \sigma$  with  $S(\rho \| \sigma) = k$ , the distance between  $\rho$  and  $\rho'$ , where  $\rho'/2^{O(k)} \leq \sigma$ , is less in the classical case than in the quantum case. More formally, the dependence on  $r$  in Theorem 2 is  $O(1/\sqrt{r})$  whereas in the classical analogue, Result 2', the dependence is like  $O(1/r)$ . The better dependence in the classical scenario enables us to prove a kind of converse to the classical substate theorem, which is outlined in the appendix. It will be interesting to see if the dependence in the quantum setting can be improved to match the classical case, enabling us to prove a similar quantum converse.

Another open question is if there is an alternate proof for the quantum substate theorem which does not go through observational divergence lifting. It will also be interesting to find yet more applications of the classical and quantum substate theorems.

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## A Relationships between three distinguishability measures

In this paper we have seen two measures of distinguishability between quantum states viz. relative entropy and observational divergence. The substate theorem gives a connection between observational divergence and a third measure of distinguishability between quantum states, which we call the *substate relative entropy*. We define three variants of substate relative entropy below, and study the relationships between them and relative entropy and observational divergence.

**Definition 11 (Substate relative entropy)** *Let  $\rho, \sigma$  be two quantum states in the same Hilbert space  $\mathcal{H}$ . The state  $\rho$  is said to have the  $k$ -substate property with respect to  $\sigma$  if for all  $r \geq 1$ , there exists a quantum state  $\rho(r)$  in  $\mathcal{H}$  such that  $\|\rho - \rho(r)\|_{\text{tr}} \leq \frac{2}{r}$  and  $\left(\frac{r-1}{r2^k}\right) \rho(r) \leq \sigma$ . The state  $\rho$  is said to have the weak  $k$ -substate property with respect to  $\sigma$  if  $\|\rho - \rho(r)\|_{\text{tr}}$  is upper bounded by  $\frac{2}{\sqrt{r}}$  instead of  $\frac{2}{r}$ . The state  $\rho$  is said to have the strong  $k$ -substate property with respect to  $\sigma$  if  $\frac{\rho}{2^k} \leq \sigma$ . The substate relative entropy of  $\rho$  with respect to  $\sigma$ ,  $S^{\text{sub}}(\rho|\sigma)$ , is defined to be the infimum over  $k$  such that  $\rho$  has the  $k$ -substate property with respect to  $\sigma$ . The quantities strong substate relative entropy,  $S^{\text{sub, strong}}$ , and weak substate relative entropy,  $S^{\text{sub, weak}}$ , are defined analogously.*

**Remark:** The classical version of strong substate relative entropy was also defined by Jain, Klauck and Nayak [JKN08] where they call it *relative co-min entropy*. The strong substate relative entropy was also defined by Renner in [Ren08] where it is called *relative min-entropy*. Our definitions of substate relative entropy and weak substate relative entropy are similar, but actually incomparable, to the *smooth relative min-entropy* defined in [Ren08].

The next proposition lists some easy properties of substate relative entropy.

**Proposition 3** *Let  $\rho, \sigma$  be density matrices in the same Hilbert space  $\mathcal{H}$ . Then,*

1.  $S^{\text{sub}}(\rho|\sigma) \geq 0$ , with equality iff  $\rho = \sigma$ ;
2.  $S^{\text{sub}}(\rho|\sigma) < \infty$  iff  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ ;

3.  $S^{sub}(\cdot\|\cdot)$  is continuous in its two arguments over the domain of pairs of full-support states;
4. (Unitary invariance) If  $U$  is a unitary transformation on  $\mathcal{H}$ ,  $S^{sub}(U\rho U^\dagger\|U\sigma U^\dagger) = S^{sub}(\rho\|\sigma)$ ;
5. (Monotonicity) Suppose  $\mathcal{K}$  is a Hilbert space, and  $\rho', \sigma'$  are extensions of  $\rho, \sigma$  in  $\mathcal{H} \otimes \mathcal{K}$ . Then,  $S^{sub}(\rho\|\sigma) \leq S^{sub}(\rho'\|\sigma')$ . This implies, via unitary invariance and the Kraus representation theorem, that if  $\mathcal{T}$  is a completely positive trace preserving superoperator from  $\mathcal{H}$  to a Hilbert space  $\mathcal{L}$ , then  $S^{sub}(\mathcal{T}\rho\|\mathcal{T}\sigma) \leq S^{sub}(\rho\|\sigma)$ .

Analogous statements hold for  $S^{sub,weak}$  and  $S^{sub,strong}$  also.

The following proposition states various relationships between our three measures of distinguishability that we have mentioned earlier.

**Proposition 4** Let  $P, Q$  be two probability distributions on  $[n]$  and  $\rho, \sigma$  be two quantum states in  $\mathbb{C}^n$ . Then,

1. (Classical substate theorem)  $S^{sub}(P\|Q) \leq D(P\|Q)$ ;
2. (Quantum substate theorem)  $S^{sub,weak}(\rho\|\sigma) \leq D(\rho\|\sigma) + 4\sqrt{D(\rho\|\sigma) + 1} + 2\log(D(\rho\|\sigma) + 1) + 5$ ;
3.  $D(\rho\|\sigma) \leq 2S^{sub}(\rho\|\sigma) + 2$ ;
4.  $S(\rho\|\sigma) \leq S^{sub,strong}(\rho\|\sigma)$ ;
5.  $D(P\|Q) - 1 \leq S(P\|Q) \leq D(P\|Q)(n - 1)$ .
6.  $D(\rho\|\sigma) - 1 \leq S(\rho\|\sigma) \leq D(\rho\|\sigma)(n - 1) + \log n$ .
7. There exist probability distributions  $P, Q$  on  $[n]$  such that  $S(P\|Q) > \left(\frac{D(P\|Q)}{2} - 1\right)(n - 2) - 1$ .
8. There exists a two-outcome POVM  $\mathcal{F}$  on  $\mathbb{C}^n$  such that  $S(\rho\|\sigma) \geq S(\mathcal{F}\rho\|\mathcal{F}\sigma) \geq \frac{S(\rho\|\sigma) - \log n}{n-1} - 1$ .

**Remarks:**

1. From Parts 1 and 3 of Proposition 4, we see that the classical substate theorem (Result 2') has a converse.
2. Unfortunately, we are unable to prove a converse to the quantum substate theorem (Result 2) as Part 2 of Proposition 4 only guarantees a weak substate property between the two quantum states  $\rho, \sigma$ .
3. Part 8 of Proposition 4 is a counterpart to monotonicity of relative entropy (Fact 4).

**Proof (Proposition 4):**

1. Define  $k := D(P\|Q)$ . Without loss of generality,  $k > 0$ . Let  $r \geq 1$ . Define the set  $\text{Bad} := \{i \in [n] : P(i)/2^{rk} > Q(i)\}$ . Then,

$$k = D(P\|Q) \geq P(\text{Bad}) \log \frac{P(\text{Bad})}{Q(\text{Bad})} > P(\text{Bad}) \cdot rk \Rightarrow P(\text{Bad}) < \frac{1}{r},$$

which is the same as expression (3) in Section 1.2. We can now argue similarly as in the proof of Result 2' to prove Part 1 of the present proposition.

2. Follows from Theorem 3 and Lemma 3.

3. Without loss of generality,  $0 < k_1 := D(\rho\|\sigma) < +\infty$ . Let  $F$  be a POVM element in  $\mathbb{C}^n$  such that

$$k_1 = p \log(p/q) \Rightarrow q = \frac{p}{2^{k_1/p}},$$

where  $p := \text{Tr}(F\rho)$  and  $q := \text{Tr}(F\sigma)$ . Note that  $p > 0$ . Let  $r := 2/p$ . Since  $\rho, \sigma$  have the  $k$ -substate property, let  $\rho'$  be the quantum state in  $\mathbb{C}^n$  such that  $\|\rho - \rho'\|_{\text{tr}} \leq \frac{2}{r} = p$  and  $(\frac{r-1}{r2^{rk}})\rho' \leq \sigma$ . Define  $p' := \text{Tr}(F\rho')$ . Then,  $p' \geq p/2$ . Also,

$$\begin{aligned} \frac{p}{2^{k_1/p}} &= q = \text{Tr}(F\sigma) \geq \left(\frac{1-r^{-1}}{2^{rk}}\right) \text{Tr}(F\rho') = \left(1 - \frac{p}{2}\right) \frac{p'}{2^{rk}} \\ &\geq \frac{p}{2^{rk+2}}. \end{aligned}$$

The last inequality above follows because  $p \leq 1$  and  $p' \geq p/2$ . This implies that

$$rk + 2 \geq \frac{k_1}{p} \Rightarrow p(rk + 2) \geq k_1 \Rightarrow 2k + 2 \geq k_1,$$

where the second implication follows because  $p \leq 1$  and  $p = 2/r$ . This completes the proof of Part 3 of the present proposition.

4. Without loss of generality,  $k < +\infty$ . We have

$$S(\rho\|\sigma) = \text{Tr} \rho \log \rho - \text{Tr} \rho \log \sigma \leq \text{Tr} \rho \log \rho - \text{Tr} \rho \log \frac{\rho}{2^k} = k \cdot \text{Tr} \rho = k.$$

The inequality above is by monotonicity of the logarithm function on positive operators [L6w34].

5. Without loss of generality,  $0 < D(P\|Q) < +\infty$ . The lower bound on  $S(P\|Q)$  was proved in Proposition 2. Define  $x_i = \log(p_i/q_i)$ . We can assume without loss of generality, by perturbing  $Q$  slightly, that the values  $x_i$  are distinct for distinct  $i$ . Let  $S' = \{i : x_i > 0\}$ . Let  $k := D(P\|Q)$ . Let For all positive  $l$ , define  $S_l := \{i \in [n] : x_i \geq l\}$ . Therefore,

$$k \geq \Pr_P[S_l] \log \frac{\Pr_P[S_l]}{\Pr_Q[S_l]} \geq \Pr_P[S_l] l \Rightarrow \Pr_P[S_l] \leq k/l.$$

Assume without loss of generality that  $x_1 < x_2 < \dots < x_n$ . Then if  $x_i > 0$ ,  $\Pr_P[S_{x_i}] \leq k/x_i$ . Since  $S(P\|Q) \leq \sum_{i \in S'} p_i x_i$ , the upper bound on  $S(P\|Q)$  is maximised when  $S' = \{2, \dots, n\}$ ,  $p_n = k/x_n$ ,  $p_i = k(1/x_i - 1/x_{i+1})$  for all  $i \in \{2, \dots, n-1\}$ , and  $p_1 = 1 - \sum_{i=2}^n p_i$ . Then,

$$\begin{aligned} S(P\|Q) &\leq \sum_{i=2}^n p_i x_i = k \sum_{i=2}^{n-1} x_i (1/x_i - 1/x_{i+1}) + k = k \sum_{i=2}^{n-1} \frac{x_{i+1} - x_i}{x_{i+1}} + k \leq k \sum_{i=2}^{n-1} 1 + k \\ &= k(n-1). \end{aligned}$$

6. Without loss of generality,  $0 < D(\rho\|\sigma) < +\infty$ . The lower bound on  $S(\rho\|\sigma)$  was proved in Proposition 2. Let us measure  $\rho$  and  $\sigma$  in the eigenbasis of  $\sigma$ . We get two distributions,  $P$  and  $Q$ . Below, we will sometimes think of  $P, Q$  as diagonal density matrices. From Part 5 of the present proposition, it follows that

$$\begin{aligned} D(P\|Q)(n-1) &\geq S(P|Q) = \text{Tr}(P \log P) - \text{Tr}(P \log Q) \geq -\log n - \text{Tr}(P \log Q) \\ &= -\log n - \text{Tr}(\rho \log \sigma) = -\log n + S(\rho\|\sigma) - \text{Tr}(\rho \log \rho) \\ &\geq -\log n + S(\rho\|\sigma). \end{aligned}$$

The second equality above holds since the measurement was in the eigenbasis of  $\sigma$ .

Thus,

$$S(\rho\|\sigma) \leq D(P\|Q)(n-1) + \log n \leq D(\rho\|\sigma)(n-1) + \log n,$$

where the second inequality is by monotonicity of observational divergence (Proposition 1).

7. Fix  $a > 1$ ,  $k > 0$ . Define for all  $i \in \{1, \dots, n-1\}$ ,  $p_i := a^{-i}(a-1)$ , and  $p_n := a^{-(n-1)}$ . Define for all  $i \in \{2, \dots, n\}$ ,  $q_i := p_i 2^{-ka^{i-1}}$ , and  $q_1 := 1 - \sum_{i=2}^n q_i$ . Define  $P := (p_1, \dots, p_n)$ ,  $Q := (q_1, \dots, q_n)$ ;  $P, Q$  are probability distributions on  $[n]$ . For any  $r > 1$ , define  $\tilde{P}$  to be the vector  $(p_1, \dots, p_{\lceil \log_a r \rceil}, 0, \dots, 0)$  normalised in order to make it a probability distribution on  $[n]$  (if  $r > a^{n-1}$ , there is no suffix of zeroes in the definition of  $\tilde{P}$ ). It is easy to see that  $\|P - \tilde{P}\|_1 \leq 2/r$  and  $\frac{(r-1)\tilde{P}}{r2^{rk}} \leq Q$ . This shows that  $S^{sub}(P\|Q) \leq k$ ; hence  $D(P\|Q) \leq 2(k+1)$  by Part 3 of the present proposition.

Now,

$$\begin{aligned} S(P\|Q) &= \sum_{i=1}^n p_i \log \frac{p_i}{q_i} \geq p_1 \log p_1 + \sum_{i=2}^n p_i \log \frac{p_i}{q_i} > -0.9 + (n-2) \frac{k(a-1)}{a} + k \\ &= k(n-1) - \frac{k(n-2)}{a} - 0.9. \end{aligned}$$

The second inequality above follows because  $p \log p > -0.9$  for all  $0 \leq p \leq 1$ . By choosing  $a$  large enough, we can achieve  $S(P\|Q) > k(n-1) - 1$ . This completes the proof of Part 7 of the present proposition.

8. The upper bound on  $S(\mathcal{F}\rho\|\mathcal{F}\sigma)$  follows from the monotonicity of relative entropy (Fact 4). Without loss of generality,  $0 < S(\rho\|\sigma) < +\infty$ . We know that there exists a POVM element  $F$  in  $\mathbb{C}^n$  such that  $D(\rho\|\sigma) = p \log(p/q)$ , where  $p := \text{Tr } F\rho$  and  $q := \text{Tr } F\sigma$ . Define the two-outcome POVM  $\mathcal{F}$  on  $\mathbb{C}^n$  to be  $(F, \mathbb{1} - F)$ , where  $\mathbb{1}$  is the identity operator on  $\mathbb{C}^n$ . Then, the probability distributions  $\mathcal{F}\rho = (p, 1-p)$  and  $\mathcal{F}\sigma = (q, 1-q)$ . Note that

$$S(\mathcal{F}\rho\|\mathcal{F}\sigma) = p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q} > p \log \frac{p}{q} - 1 = D(\rho\|\sigma) - 1,$$

where the inequality follows because  $x \log x > -1$  for all  $0 \leq x \leq 1$ . From Part 6 of the present proposition, it follows that

$$\begin{aligned} S(\rho\|\sigma) &\leq D(\rho\|\sigma)(n-1) + \log n \leq (S(\mathcal{F}\rho\|\mathcal{F}\sigma) + 1)(n-1) + \log n \\ \Rightarrow S(\mathcal{F}\rho\|\mathcal{F}\sigma) &\geq \frac{S(\rho\|\sigma) - \log n}{n-1} - 1. \end{aligned}$$

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