A Super-Additivity Inequality for Channel Capacity of Classical-Quantum Channels

Rahul Jain *
National University of Singapore

Abstract

We show a super-additivity inequality for the *channel capacity* of *classical-quantum* $(\mathsf{c}-\mathsf{q})$ channels.

1 Introduction

Let X be a finite set and $\mathcal{S}$ be the set of all quantum states. A classical-quantum $(\mathsf{c}-\mathsf{q})$ channel E is a map from X to $\mathcal{S}$. All the channels we consider will be $\mathsf{c}-\mathsf{q}$ channels and we would avoid mentioning $\mathsf{c}-\mathsf{q}$ explicitly from now on. For a probability distribution μ over X, let E_μ be the bipartite state $\mathbb{E}_{x\leftarrow\mu}[|x\rangle\langle x|\otimes E(x)]$. Let $I(E_\mu)$ be the *mutual information*[1] between the two systems in E_μ. The channel capacity of such a channel is defined as follows.

Definition 1.1 (Channel capacity) *Channel capacity of the channel $E : X \mapsto \mathcal{S}$ is defined as $C(E) \stackrel{def}{=} \max_\mu I(E_\mu)$.*

2 Our result

We show a super-additivity inequality for channel capacity of a channel. Before describing our result we need to define the notion of a *derived channel*.

Definition 2.1 (Derived channel) *Let X and Y be finite sets and let $E : X \times Y \rightarrow \mathcal{S}$ be a channel. For a collection $\{\mu_x : x \in X\}$, where each μ_x is a probability distribution on Y, let $F : X \rightarrow \mathcal{S}$ be a channel given by $F(x) \stackrel{def}{=} \mathbb{E}_{y\leftarrow\mu_x}[E(x,y)]$. Such a channel F is referred to as an E-derived channel on X. Similarly we can define E-derived channels on Y using collections of probability distributions on X.*

Now we can state our result.

*Centre for Quantum Technologies and Department of Computer Science, 3 Science Drive 2, Singapore 117543. Email: rahul@comp.nus.edu.sg. This work was done while the author was at U.C Berkeley where it was supported by an Army Research Office (ARO), North California, grant number DAAD 19-03-1-00082.

[1]For a bipartite system AB, the mutual information between the two systems is defined as $I(A:B) \stackrel{def}{=} S(A) + S(B) - S(AB)$, where $S(\cdot)$ represents the *von-Neumann entropy*.

Theorem 2.1 (Super-additivity) *Let k be a positive integer. Let $X_1, X_2, \ldots, X_k$ be finite sets. Let $E : X_1 \times X_2 \ldots \times X_k \to \mathcal{S}$ be a channel. For $i \in [k]$, let $\mathcal{C}_i$ be the set of all E-derived channels on X_i. Then,*

$$C(E) \quad \geq \quad \sum_{i=1}^{k} \min_{F_i \in \mathcal{C}_i} C(F_i) \ .$$

Proof: We show the result for $k = 2$ and for larger k it follows automatically. Let $k = 2$, $X \stackrel{def}{=} X_1$ and $Y \stackrel{def}{=} X_2$. For each $x \in X$, let $E^x : Y \to \mathcal{S}$ be an E-derived channel on Y given by $E^x(y) \stackrel{def}{=} E(x, y)$. For each $x \in X$, let μ_x be a probability distribution on Y such that $I(E^x_{\mu_x}) = C(E^x)$. Now let $E^X : X \to \mathcal{S}$ be an E-derived channel on X given by $E^X(x) \stackrel{def}{=} \mathbb{E}_{y \leftarrow \mu_x}[E(x, y)]$. Let μ_X be a distribution on X such that $I(E^X_{\mu_X}) = C(E^X)$. Let μ be the distribution on $X \times Y$ arising by sampling from X according to μ_X, and conditioned on $X = x$, sampling from Y according to μ_x. Now the following *chain rule property* holds for mutual information.

Fact 2.1 *Let X, Y, Z be a tripartite system where X is a classical system. Let P be the distribution of X. Then,*

$$I(XY : Z) = I(X : Z) + \mathbb{E}_{x \leftarrow P}[I((Y : Z) \mid X = x)] \ .$$

Now we have,

$$\begin{aligned}
C(E) &\geq I(E_\mu) \quad \text{(from definition of capacity)} \\
&= I(E^X_{\mu_X}) + \mathbb{E}_{x \leftarrow \mu_X}[I(E^x_{\mu_x})] \quad \text{(from chain rule for mutual information)} \\
&= C(E^X) + \mathbb{E}_{x \leftarrow \mu_X}[C(E^x)] \\
&\geq \min_{F_1 \in \mathcal{C}_1} C(F_1) + \min_{F_2 \in \mathcal{C}_2} C(F_2) \ .
\end{aligned}$$

This finishes our proof. ■