Teleportation of Quantum States (1993; Bennett, Brassard, Crepeau, Jozsa, Peres, Wootters)

Rahul Jain
U. Waterloo and Institute for Quantum Computing, rjain@cs.uwaterloo.ca

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1 Problem definition

An *n-qubit quantum state* is a positive semi-definite operator of unit trace in the complex Hilbert space $\mathbb{C}^{2^n}$. A *pure* quantum state is a quantum state with a unique non-zero eigenvalue. A pure state is also often represented by the unique unit eigenvector corresponding to the unique non-zero eigenvalue. In this article the standard (ket, bra) notation is followed as is often used in quantum mechanics, in which $|v\rangle$ (called as 'ket v') represents a column vector and $\langle v|$ (called as 'bra v') represents its conjugate transpose. A *classical n-bit* state is simply a probability distribution on the set $\{0,1\}^n$.

Let $\{|0\rangle, |1\rangle\}$ be the standard basis for $\mathbb{C}^2$. For simplicity of notation $|0\rangle \otimes |0\rangle$ are represented as $|0\rangle|0\rangle$ or simply $|00\rangle$. Similarly $|0\rangle\langle 0|$ represents $|0\rangle \otimes \langle 0|$. An EPR pair is a special two-qubit quantum state defined as $|\psi\rangle \triangleq \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. It is one of the four *Bell states* which form a basis for $\mathbb{C}^4$.

Suppose there are two spatially separated parties Alice and Bob and Alice wants to send an arbitrary n-qubit quantum state ρ to Bob. Since classical communication is much more reliable, and possibly cheaper, than quantum communication, it is desirable that this task be achieved by communicating just classical bits. Such a procedure is referred to as *teleportation*.

Unfortunately, it is easy to argue that this is in fact not possible if arbitrary quantum states need to be communicated faithfully. However Bennett, Brassard, Crepeau, Jozsa, Peres, Wootters [2] presented the following nice solution to it.

2 Key results

Alice and Bob are said to share an EPR pair if each hold one qubit of the pair. In this article a standard notation is followed in which classical bits are called 'cbits' and shared EPR pairs are called 'ebits'. Bennett et al. showed the following:

Theorem 2.1 *Teleportation of an arbitrary n-qubit state can be achieved with $2n$ cbits and n ebits.*

These shared EPR pairs are referred to as *prior entanglement* to the protocol since they are shared at the beginning of the protocol (before Alice gets her input state) and are independent of Alice's input state. This solution is a good compromise since it is conceivable that Alice and Bob share several EPR pairs at the beginning, when they are possibly together, in which case they do not require a quantum channel. Later they can use these EPR pairs to transfer several quantum states when they are spatially separated.

Let us now see how Bennett el al. [2] achieve teleportation. Let us first note that in order to show Theorem 2.1 it is enough to show that a single qubit, which is possibly a part of a larger state ρ can be teleported, while preserving its entanglement with the rest of the qubits of ρ, using 2 cbits and 1 ebit. Let us also note that the larger state ρ can now be assumed to be a pure state without loss of generality.

Theorem 2.2 *Let $|\phi\rangle_{AB} = a_0|\phi_0\rangle_{AB}|0\rangle_A + a_1|\phi_1\rangle_{AB}|1\rangle_A$, where a_0, a_1 are complex numbers with $|a_0|^2 + |a_1|^2 = 1$. Subscripts A, B (representing Alice and Bob respectively) on qubits signify their owner.*

It is possible for Alice to send two classical bits to Bob such that at the end of the protocol the final state is $a_0|\phi_0\rangle_{AB}|0\rangle_B + a_1|\phi_1\rangle_{AB}|1\rangle_B$.

Proof: For simplicity of notation, let us assume below that $|\phi_0\rangle_{AB}$ and $|\phi_1\rangle_{AB}$ do not exist. The proof is easily modified when they do exist by tagging them along. Let an EPR pair $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$ be shared between Alice and Bob. Let us refer to the qubit under concern that needs to be teleported as the input qubit.

The combined starting state of all the qubits is

$$\begin{aligned} |\theta_0\rangle_{AB} &= |\phi\rangle_{AB}|\psi\rangle_{AB} \\ &= (a_0|0\rangle_A + a_1|1\rangle_A)(\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)) \end{aligned}$$

Let CNOT (*controlled-not*) gate be a two-qubit unitary operation described by the operator $|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$. Alice now performs a CNOT gate on the input qubit and her part of the shared EPR pair. The resulting state is then,

$$|\theta_1\rangle_{AB} = \frac{a_0}{\sqrt{2}}|0\rangle_A(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) + \frac{a_1}{\sqrt{2}}|1\rangle_A(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)$$

Let the Hadamard transform be a single qubit unitary operation with operator $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1|$. Alice next performs a Hadamard transform on her input qubit. The resulting state then is,

$$\begin{aligned} |\theta_2\rangle_{AB} &= \frac{a_0}{2}(|0\rangle_A + |1\rangle_A)(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) + \frac{a_1}{2}(|0\rangle_A - |1\rangle_A)(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B) \\ &= \frac{1}{2}(|00\rangle_A(a_0|0\rangle_B + a_1|1\rangle_B) + |01\rangle_A(a_0|1\rangle_B + a_1|0\rangle_B)) \\ &\quad + \frac{1}{2}(|10\rangle_A(a_0|0\rangle_B - a_1|1\rangle_B) + |11\rangle_A(a_0|1\rangle_B - a_1|0\rangle_B)) \end{aligned}$$

Alice next measures the two qubits in her possession in the standard basis for $\mathbb{C}^4$ and sends the result of the measurement to Bob.

Let the four Pauli gates be the single qubit unitary operations: Identity: $P_{00} = |0\rangle\langle 0| + |1\rangle\langle 1|$, bit flip: $P_{01} = |1\rangle\langle 0|+|0\rangle\langle 1|$, phase flip: $P_{10} = |0\rangle\langle 0|-|1\rangle\langle 1|$ and bit flip together with phase flip: $P_{11} = |1\rangle\langle 0|-|0\rangle\langle 1|$. On receiving the two bits c_0c_1 from Alice, Bob performs the Pauli gate $P_{c_0c_1}$ on his qubit. It is now easily verified that the resulting state of the qubit with Bob would be $a_0|0\rangle_B + a_1|1\rangle_B$. The input qubit is successfully teleported from Alice to Bob ! Please refer to Figure 1 for the overall protocol. ■

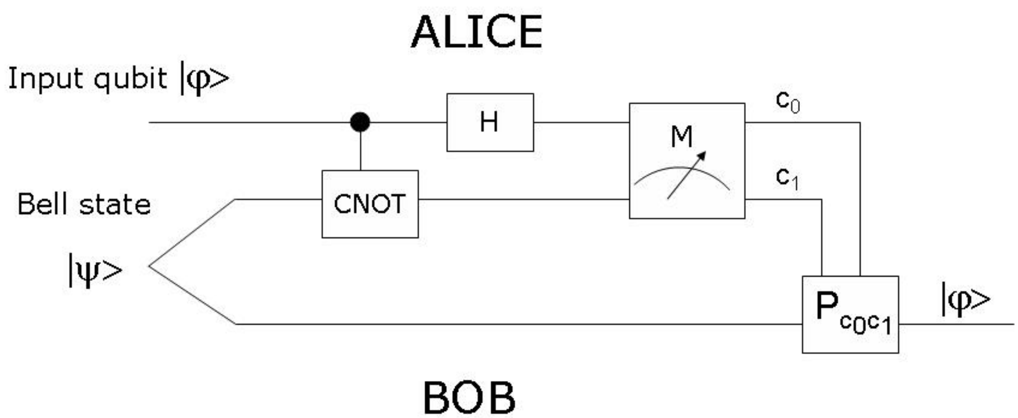

Figure 1: Teleportation protocol. H represent Hadamard transform and M represents measurement in the standard basis for $\mathbb{C}^4$.

2.1 Super-dense coding

Super-dense coding [11] protocol is a dual to the teleportation protocol. In this Alice transmits 2 cbits of information to Bob using 1 qubit of communication and 1 shared ebit. It is discussed more elaborately in another article in the encyclopedia.

2.2 Lower bounds on resources

The above implementation of teleportation requires 2 cbits and 1 ebit for teleporting 1 qubit. It was argued in [2] that these resource requirements are also independently optimal. That is 2 cbits need to be communicated to teleport a qubit independent of how many ebits are used. Also 1 ebit is required to teleport one qubit independent of how much (possibly two-way) communication is used.

2.3 Remote state preparation

Closely related to the problem of teleportation is the problem of *Remote state preparation* (RSP) introduced by Lo [10]. In teleportation Alice is just given the state to be teleported in some input register and has no other information about it. In contrast, in RSP, Alice knows a *complete description* of the input state that needs to be teleported. Also in RSP, Alice is not required to maintain any correlation of the input state with the other parts of a possibly larger state as is achieved in teleportation. The extra knowledge that Alice possesses about the input state can be used to devise protocols for probabilistically exact RSP with one cbit and one ebit per qubit asymptotically [3]. In a probabilistically exact RSP, Alice and Bob can abort the protocol with a small probability, however when they do not abort, the state produced with Bob at the end of the protocol, is exactly the state that Alice intends to send.

2.4 Teleportation as a private quantum channel

The teleportation protocol that has been discussed in this article also satisfies an interesting privacy property. That is if there was a third party, say Eve, having access to the communication channel between Alice and Bob, then Eve learns nothing about the input state of Alice that she is teleporting to Bob. This is because the distribution of the classical messages of Alice is always uniform, independent of her input state. Such a channel is referred to as a *Private quantum channel* [6, 1, 8].

3 Applications

Apart from the main application of transporting quantum states over large distances using only classical channel, the teleportation protocol finds other important uses as well. A generalization of this protocol to implement unitary operations [7], is used in *Fault tolerant computation* in order to construct an infinite class of fault tolerant gates in a uniform fashion. In another application, a form of teleportation called as the error correcting teleportation, introduced by Knill [9], is used in devising quantum circuits that are resistant to very high levels of noise.

4 Experimental results

Teleportation protocol has been experimentally realized in various different forms. To name a few, by Boschi et al. [4] using optical techniques, by Bouwmeester et al. [5] using photon polarization, by Nielsen et al. [12] using *Nuclear magnetic resonance* (NMR) and by Ursin et al. [13] using photons for long distance.

5 Cross references

Entry on *Super-dense coding.*

Recommended reading

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