
Differential Privacy Dynamics of Langevin Diffusion and Noisy Gradient Descent

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Abstract

What is the information leakage of an iterative randomized learning algorithm about its training data, when the internal state of the algorithm is *private*? How much is the contribution of each specific training epoch to the information leakage through the released model? We study this problem for noisy gradient descent algorithms, and model the *dynamics* of Rényi differential privacy loss throughout the training process. Our analysis traces a provably *tight* bound on the Rényi divergence between the pair of probability distributions over parameters of models trained on neighboring datasets. We prove that the privacy loss converges exponentially fast, for smooth and strongly convex loss functions, which is a significant improvement over composition theorems (which over-estimate the privacy loss by upper-bounding its total value over all intermediate gradient computations). For Lipschitz, smooth, and strongly convex loss functions, we prove optimal utility with a small gradient complexity for noisy gradient descent algorithms.

1 Introduction

Machine learning models leak a significant amount of information about their training data, through their parameters and predictions [21, 18, 5]. Iterative randomized training algorithms can limit this information leakage and bound the differential privacy loss of the learning process [3, 1, 8, 9]. The strength of this certified defense is determined by an *upper bound* on the (Rényi) divergence between the probability distributions of model parameters learned on any pair of neighboring datasets.

The general method to compute the differential privacy bound for gradient perturbation-based learning algorithms is to view the process as a number of (identical) differential privacy mechanisms, and to compute the *composition* of their bounds. However, this over-estimates the privacy loss of the released model [13, 19], and results in a loose differential privacy bound. This is because composition bounds also accounts for the leakage of all intermediate gradient updates, even though only the final model parameters are observable to adversary. Feldman et al. [8, 9] address this issue for the privacy analysis of gradient computations over *one single* training epoch, for smooth and convex loss functions. However, in learning a model over multiple training epochs, such a guarantee is quantitatively similar to composition bounds of privacy amplification by sub-sampling [8]. The open challenge, that we tackle in this paper, is to provide an analysis that can tightly bound the privacy loss of the *released model* after K training epochs, for any K .

We present a novel analysis for privacy dynamics of noisy gradient descent with smooth and strongly convex loss functions. We construct a pair of continuous-time Langevin diffusion [20] processes that trace the probability distributions over the model parameters of noisy GD. Subsequently, we derive differential inequalities bounding the *rate* of privacy loss (worst case Rényi divergence between the **coupled stochastic processes** associated with neighboring datasets) throughout the training process.

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We then prove an exponentially-fast converging privacy bound for noisy GD: (simplified theorem) Under 1-strongly convex and β -smooth loss function $\ell(\theta; \mathbf{x})$ with total gradient sensitivity 1, the noisy GD Algorithm 1, with initial parameter vector $\theta_0 \sim \Pi_{\mathcal{C}}(\mathcal{N}(0, 2\sigma^2\mathbb{I}_d))$ and step size $\eta < \frac{1}{\beta}$, satisfies (α, ε) -Rényi DP with $\varepsilon = O\left(\frac{\alpha}{\sigma^2 n^2}(1 - e^{-\eta\frac{K}{2}})\right)$, where n is the size of the training set.

This guarantee shows that the privacy loss **converges** exponentially in the number of iterations K , instead of growing proportionally with K as in the composition-based analysis of the same algorithms. Our bound captures a strong privacy amplification due to the dynamics (and convergence) of differential privacy over the noisy gradient descent algorithm with private internal state.

We analyze the *tightness* of the bound, the *utility* of the randomized algorithm under the computed differential privacy bound, as well as its *gradient complexity* (number of required gradient computations). We prove the **tightness guarantee** for our bound by showing that there exist a loss function and neighboring datasets such that the divergence between corresponding model parameter distributions matches our privacy bound. For Lipschitz, smooth, and strongly convex loss functions, we prove that noisy GD achieves **optimal utility** under differential privacy with an error of order $O(\frac{d}{n^2\varepsilon^2})$, with a *small gradient complexity* of order $O(n \log(n))$. This improves over the prior utility results for noisy SGD algorithms [3]. Our analysis results in a significantly smaller gradient complexity by a factor of $n/\log(n)$, and a slightly better utility by a factor of polylog(n).

We anticipate that our work will have a positive societal impact, by paving the way for building accurate and privacy preserving machine learning systems for sensitive personal data.

2 Preliminaries on differential privacy

Let \mathcal{X} be the data universe, and a dataset D contain n records from it: $D = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \in \mathcal{X}^n$. We refer to a dataset pair D, D' as *neighboring* if they differ in one data record. A measure ν is said to be absolutely continuous with respect to another measure ν' on same space (denoted as $\nu \ll \nu'$) if for all measurable set S , $\nu(S) = 0$ whenever $\nu'(S) = 0$.

Definition 2.1 ([17] Rényi differential privacy). *Let $\alpha > 1$. A randomized algorithm $\mathcal{A} : \mathcal{X}^n \rightarrow \mathbb{R}^d$ satisfies (α, ε) -Rényi Differential Privacy (RDP), if for any two neighboring datasets $D, D' \in \mathcal{X}^n$, the α Rényi divergence $R_\alpha(\mathcal{A}(D) \parallel \mathcal{A}(D')) \leq \varepsilon$. For a pair of measures ν, ν' over the same space with $\nu \ll \nu'$, $R_\alpha(\nu \parallel \nu')$ is defined as*

$$R_\alpha(\nu \parallel \nu') = \frac{1}{\alpha - 1} \log E_\alpha(\nu \parallel \nu'), \quad \text{where} \quad E_\alpha(\nu \parallel \nu') = \int \left(\frac{d\nu}{d\nu'}\right)^\alpha d\nu'. \quad (1)$$

We refer to $R_\alpha(\mathcal{A}(D) \parallel \mathcal{A}(D'))$ also as the *Rényi privacy loss* of algorithm \mathcal{A} on datasets D, D' . An RDP guarantee can be converted to (ε, δ) -DP guarantee [17, Proposition 5].

Definition 2.2 ([22] Rényi information). *Let $\alpha > 1$. For any two measures ν, ν' over \mathbb{R}^d with $\nu \ll \nu'$ and corresponding probability density functions p, p' , if $\frac{p(\theta)}{p'(\theta)}$ is differentiable, the α -Rényi Information of ν with respect to ν' is*

$$I_\alpha(\nu \parallel \nu') = \frac{4}{\alpha^2} \mathbb{E}_{\theta \sim \nu'} \left[\left\| \nabla \frac{p(\theta)^{\frac{\alpha}{2}}}{p'(\theta)^{\frac{\alpha}{2}}} \right\|_2^2 \right] = \mathbb{E}_{\theta \sim \nu'} \left[\frac{p(\theta)^{\alpha-2}}{p'(\theta)^{\alpha-2}} \left\| \nabla \frac{p(\theta)}{p'(\theta)} \right\|_2^2 \right]. \quad (2)$$

See the Appendix B for a comprehensive presentation of preliminaries.

3 Privacy analysis of noisy gradient descent

Let $D = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ be a dataset of size n with records taken from a universe \mathcal{X} . For a given machine learning algorithm, let $\ell(\theta; \mathbf{x}) : \mathcal{X} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a loss function of a parameter vector $\theta \in \mathcal{C}$ on the data point \mathbf{x} , where \mathcal{C} is a closed convex set (can be \mathbb{R}^d).

A generic formulation of the optimization problem to learn the model parameters, is in the form of empirical risk minimization (ERM) with the following objective, where $\mathcal{L}_D(\theta)$ is the empirical loss

of the model, with parameter vector θ , on a dataset D .

$$\theta^* = \arg \min_{\theta \in \mathcal{C}} \mathcal{L}_D(\theta), \quad \text{where} \quad \mathcal{L}_D(\theta) = \frac{1}{n} \sum_{\mathbf{x} \in D} \ell(\theta; \mathbf{x}). \quad (3)$$

Releasing this optimization output (i.e., θ^*) can leak information about the dataset D , hence violating data privacy. To mitigate this risk, there exist randomized algorithms to ensure that the (α -Rényi) privacy loss of the ERM algorithm is upper-bounded by ε , i.e., the algorithm satisfies (α, ε) -RDP.

Algorithm 1 $\mathcal{A}_{\text{Noisy-GD}}$: Noisy Gradient Descent

Input: Dataset $D = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$, loss function $\ell(\theta; \mathbf{x})$, closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$, learning rate η , noise variance σ^2 , initial parameter vector θ_0 .

- 1: **for** $k = 0, 1, \dots, K - 1$ **do**
 - 2: $g(\theta_k; D) = \sum_{i=1}^n \nabla \ell(\theta_k; \mathbf{x}_i)$
 - 3: $\theta_{k+1} = \Pi_{\mathcal{C}}(\theta_k - \frac{\eta}{n} g(\theta_k; D) + \sqrt{2\eta} \mathcal{N}(0, \sigma^2 \mathbb{I}_d))$
 - 4: **Output** θ_K
-

In this paper, our objective is to analyze privacy loss of *Noisy Gradient Descent* (Algorithm 1), which is a randomized ERM algorithm. Let θ_k, θ'_k be the parameter vectors at the k 'th iteration of $\mathcal{A}_{\text{Noisy-GD}}$ on neighboring datasets D and D' , respectively. We denote by $\Theta_{\eta k}$ and $\Theta'_{\eta k}$ the corresponding random variables that model θ_k and θ'_k . We abuse notation to also denote their probability distributions by $\Theta_{\eta k}$ and $\Theta'_{\eta k}$. In this paper, our objective is to model and analyze the **dynamics of differential privacy** of this algorithm. More precisely, we focus on the following.

1. Compute an RDP bound (i.e., the worst case Rényi divergence $R_\alpha(\Theta_K \| \Theta'_K)$ between the output distributions of two neighboring datasets) for Algorithm 1, and analyze its tightness.
2. Compute the contribution of each iteration to the privacy loss. As we go from step $k = 1$ to K in Algorithm 1, we investigate how the algorithm's privacy loss changes as it runs the k 'th iteration (computed as $R_\alpha(\Theta_{\eta k} \| \Theta'_{\eta k}) - R_\alpha(\Theta_{\eta(k-1)} \| \Theta'_{\eta(k-1)})$).

In the end, we aim to provide a RDP bound that is tight, thus facilitating optimal utility [3]. We emphasize that our goal is to construct a theoretical framework for analyzing privacy loss of releasing the output θ_K of the algorithm, assuming *private* internal states (i.e., $\theta_1, \dots, \theta_{K-1}$).

3.1 Tracing diffusion for Noisy GD

To analyze the privacy loss of Noisy GD, which is a *discrete-time stochastic process*, we first interpolate each discrete update from θ_k to θ_{k+1} with a piece-wise continuously differentiable diffusion process. Let D and D' be a pair of arbitrarily chosen neighboring datasets. Given step-size η and initial parameter vector $\theta_0 = \theta'_0$, the respective k 'th discrete updates in Algorithm 1 on neighboring datasets D and D' are

$$\begin{cases} \theta_{k+1} = \Pi_{\mathcal{C}}(\theta_k - \eta \nabla \mathcal{L}_D(\theta_k) + \sqrt{2\eta\sigma^2} \mathbf{Z}_k), \\ \theta'_{k+1} = \Pi_{\mathcal{C}}(\theta'_k - \eta \nabla \mathcal{L}_{D'}(\theta'_k) + \sqrt{2\eta\sigma^2} \mathbf{Z}_k), \end{cases} \quad \text{with} \quad \mathbf{Z}_k \sim \mathcal{N}(0, \mathbb{I}_d). \quad (4)$$

These two discrete jumps can be interpolated with two stochastic processes Θ_t and Θ'_t over time $\eta k \leq t \leq \eta(k+1)$ respectively. At the start of each step, $t = \eta k$, the random variables $\Theta_{\eta k}$ and $\Theta'_{\eta k}$ model the distribution of the θ_k and θ'_k in the noisy GD processes respectively. During time $\eta k < t < \eta(k+1)$, we model the respective gradient updates on D and D' with the following stochastic processes.

$$\begin{cases} \Theta_t = \Theta_{\eta k} - \eta \cdot U_1(\Theta_{\eta k}) - (t - \eta k) \cdot U_2(\Theta_{\eta k}) + \sqrt{2(t - \eta k)\sigma^2} \mathbf{Z}_k \\ \Theta'_t = \Theta'_{\eta k} - \eta \cdot U_1(\Theta'_{\eta k}) + (t - \eta k) \cdot U_2(\Theta'_{\eta k}) + \sqrt{2(t - \eta k)\sigma^2} \mathbf{Z}_k \end{cases} \quad (5)$$

where the vectors $U_1(\theta) = \frac{1}{2}(\nabla \mathcal{L}_D(\theta) + \nabla \mathcal{L}_{D'}(\theta))$ and $U_2(\theta) = \frac{1}{2}(\nabla \mathcal{L}_D(\theta) - \nabla \mathcal{L}_{D'}(\theta))$ represent the average and difference between gradients on neighboring datasets D and D' respectively.

At the end of step, i.e. at $t \rightarrow \eta(k+1)$, we project Θ_t and Θ'_t onto convex set \mathcal{C} , and obtain

$$\Theta_{\eta(k+1)} = \Pi_{\mathcal{C}} \left(\lim_{t \rightarrow \eta(k+1)^-} \Theta_t \right), \Theta'_{\eta(k+1)} = \Pi_{\mathcal{C}} \left(\lim_{t \rightarrow \eta(k+1)^-} \Theta'_t \right). \quad (6)$$

By plugging (5) into (6), we compute that the projected random variable $\Theta_{\eta(k+1)}$ and $\Theta'_{\eta(k+1)}$ have the same distributions as the parameters θ_{k+1} and θ'_{k+1} at $k+1$ th step of noisy GD respectively. Repeating the construction for $k = 0, \dots, K-1$, we define two piece-wise continuous diffusion processes $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ whose distributions at time $t = \eta k$ are consistent with θ_k and θ'_k in the noisy GD processes (4) for any $k \in \{0, \dots, K-1\}$.

Definition 3.1 (Coupled tracing diffusions). *Let $\Theta_0 = \Theta'_0$ be two identically distributed random variables. We refer to the stochastic processes $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ that evolve along diffusion processes (5) in $\eta k < t < \eta(k+1)$ and undergo projection steps (6) at the end of step $t = \eta(k+1)$, as coupled tracing diffusions for noisy GD on neighboring datasets D, D' .*

The Rényi divergence $R_\alpha(\Theta_{\eta K} \| \Theta'_{\eta K})$ reflects the Rényi privacy loss of Algorithm 1 with K steps. Conditioned on observing θ_k and θ'_k , the processes $\{\Theta_t\}_{\eta k < t < \eta(k+1)}$ and $\{\Theta'_t\}_{\eta k < t < \eta(k+1)}$ in (5) are Langevin diffusions along vector fields $-U_2(\theta_k)$ and $U_2(\theta'_k)$ respectively, for duration η . Therefore, conditioned on observing θ_k and θ'_k , the diffusion processes in (5) have the following stochastic differential equations (SDEs) respectively.

$$d\Theta_t = -U_2(\theta_k)dt + \sqrt{2\sigma^2}d\mathbf{W}_t, \quad d\Theta'_t = U_2(\theta'_k)dt + \sqrt{2\sigma^2}d\mathbf{W}_t, \quad (7)$$

where $d\mathbf{W}_t \sim \sqrt{dt}\mathcal{N}(0, \mathbb{I}_d)$ describe the Wiener processes on \mathbb{R}^d . Therefore, the conditional probability density functions $p_{t|\eta k}(\theta|\theta_k)$ and $p'_{t|\eta k}(\theta|\theta'_k)$ follow the following Fokker-Planck equation. For brevity, we use $p_{t|\eta k}(\theta|\theta_k)$ and $p'_{t|\eta k}(\theta|\theta'_k)$ to represent the conditional probability density function $p(\Theta_t = \theta | \Theta_{\eta k} = \theta_k)$ and $p(\Theta'_t = \theta | \Theta'_{\eta k} = \theta'_k)$ respectively.

$$\begin{cases} \frac{\partial p_{t|\eta k}(\theta|\theta_k)}{\partial t} = \nabla \cdot (p_{t|\eta k}(\theta|\theta_k)U_2(\theta_k)) + \sigma^2 \Delta p_{t|\eta k}(\theta|\theta_k) \\ \frac{\partial p'_{t|\eta k}(\theta|\theta'_k)}{\partial t} = -\nabla \cdot (p'_{t|\eta k}(\theta|\theta'_k)U_2(\theta'_k)) + \sigma^2 \Delta p'_{t|\eta k}(\theta|\theta'_k) \end{cases} \quad (8)$$

By taking expectations over probability density function $p_{\eta k}(\theta_k)$ or $p'_{\eta k}(\theta'_k)$ on both sides of (8), we obtain the partial differential equation that models the evolution of (unconditioned) probability density function $p_t(\theta)$ and $p'_t(\theta)$ in the coupled tracing diffusions.

Lemma 1. *For coupled tracing diffusion processes (5) in time $\eta k < t < \eta(k+1)$, the equivalent Fokker-Planck equations are*

$$\begin{cases} \frac{\partial p_t(\theta)}{\partial t} = \nabla \cdot (p_t(\theta)V_t(\theta)) + \sigma^2 \Delta p_t(\theta) \\ \frac{\partial p'_t(\theta)}{\partial t} = \nabla \cdot (p'_t(\theta)V'_t(\theta)) + \sigma^2 \Delta p'_t(\theta), \end{cases} \quad (9)$$

where $V_t(\theta) = \mathbb{E}_{\theta_k \sim p_{\eta k|t}} [U_2(\theta_k)|\theta]$ and $V'_t(\theta) = \mathbb{E}_{\theta'_k \sim p'_{\eta k|t}} [-U_2(\theta_k)|\theta]$ are time-dependent vector fields on \mathbb{R}^d , and $U_2(\theta) = \frac{1}{2} [\nabla \mathcal{L}_D(\theta) - \nabla \mathcal{L}_{D'}(\theta)]$ is the difference between gradients on neighboring datasets.

By this density evolution equation, we model the noisy gradient descent updates with coupled tracing diffusions. The tracing diffusion process is similar to Langevin diffusion. Therefore, we first study the privacy dynamics in coupled tracing (Langevin) diffusions.

3.2 Privacy erosion in tracing (Langevin) diffusion

The Rényi divergence (privacy loss) $R_\alpha(\Theta_t \| \Theta'_t)$ between coupled tracing diffusion processes increases over time, as the vector fields V_t, V'_t underlying two processes are different. We refer to this phenomenon as **privacy erosion**. This increase is determined by the amount of change in the probability density functions for coupled tracing diffusions, characterized by the Fokker-Planck equations (47) for diffusions under different vector fields.

Using equation (47), we compute a bound on the rate (partial derivative) of $R_\alpha(\Theta_t \| \Theta'_t)$ over time in the following lemma, to model privacy erosion between two different diffusion processes. We refer to *coupled diffusions* as respective diffusion processes under different vector fields V_t and V'_t .

Lemma 2 (Rate of Rényi privacy loss). *Let V_t and V'_t be two vector fields on \mathbb{R}^d corresponding to a pair of arbitrarily chosen neighboring datasets D and D' with $\max_{\theta \in \mathbb{R}^d} \|V_t(\theta) - V'_t(\theta)\|_2 \leq S_v$ for all $t \geq 0$. Then, for corresponding coupled diffusions $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ under vector fields V_t and V'_t and noise variance σ^2 , the Rényi privacy loss rate at any $t \geq 0$ is upper bounded by*

$$\frac{\partial R_\alpha(\Theta_t \| \Theta'_t)}{\partial t} \leq \frac{1}{\gamma} \frac{\alpha S_v^2}{4\sigma^2} - (1 - \gamma)\sigma^2 \alpha \frac{I_\alpha(\Theta_t \| \Theta'_t)}{E_\alpha(\Theta_t \| \Theta'_t)}. \quad (10)$$

where $\gamma > 0$ is a tuning parameter that we can fix arbitrarily according to our need.

Although this lemma bounds the Rényi privacy loss rate, the term $I_\alpha(\Theta_t \| \Theta'_t)$ depends on unknown distributions Θ_t, Θ'_t , and is intractable to compute. Even with explicit expressions for distributions Θ_t, Θ'_t , the calculation would involve integration in \mathbb{R}^d which is computationally prohibitive for large d . Note that, however, the ratio I_α/E_α is always positive by definition. Therefore, the Rényi divergence (privacy loss) rate in (10) is bounded by its first component (a constant) given any fixed α .

Theorem 1 (Linear Rényi divergence bound). *Let V_t and V'_t be two vector fields on \mathbb{R}^d , with $\max_{\theta \in \mathbb{R}^d} \|V_t(\theta) - V'_t(\theta)\|_2 \leq S_v$ for all $t \geq 0$. Then, the coupled diffusions under vector fields V_t and V'_t with noise variance σ^2 for time T has α -Rényi divergence bounded by $\varepsilon = \frac{\alpha S_v^2 T}{4\sigma^2}$.*

When the vector fields are $V_t = -\nabla \mathcal{L}_D$ and $V'_t = -\nabla \mathcal{L}_{D'}$, the coupled diffusions follow Langevin diffusion. By definition B.10 of total gradient sensitivity, $\max_{\theta \in \mathbb{R}^d} \|\nabla \mathcal{L}_D(\theta) - \nabla \mathcal{L}_{D'}(\theta)\|_2 \leq \frac{S_g}{n}$.

Therefore, this naïve privacy analysis gives linear RDP guarantee for Langevin diffusion, which resembles the moment accountant analysis [1]. However, a tighter bound of Rényi privacy loss is possible with finer control of the ratio $I_\alpha(\Theta_t \| \Theta'_t)/E_\alpha(\Theta_t \| \Theta'_t)$, which by definition depends on the likelihood ratio between Θ_t and Θ'_t , thus is connected with Rényi privacy loss itself. When this ratio grows, the Rényi privacy loss rate decreases, thus slowing down privacy loss accumulation, and leading to tighter privacy bound.

Controlling Rényi privacy loss rate under isoperimetry We control the I_α/E_α term in lemma 2 by making an *isoperimetric* assumption known as *log-Sobolev inequality* [2], described as follows.

Definition 3.2 ([12] Log-Sobolev Inequality (c-LSI)). *Distribution of a random variable Θ on \mathbb{R}^d satisfies logarithmic Sobolev inequality with parameter $c > 0$, i.e. it is c-LSI, if for all functions f in the function set $\mathcal{F}_\Theta = \{f : \mathbb{R}^d \rightarrow \mathbb{R} \mid \nabla f \text{ is continuous, and } \mathbb{E}(f(\Theta)^2) < \infty\}$, we have*

$$\mathbb{E}[f(\Theta)^2 \log f(\Theta)^2] - \mathbb{E}[f(\Theta)^2] \log \mathbb{E}[f(\Theta)^2] \leq \frac{2}{c} \mathbb{E}[\|\nabla f(\Theta)\|_2^2]. \quad (11)$$

LSI was introduced by Gross [12] as a necessary and sufficient condition for rapid convergence of a diffusion processes. Recently, Vempala and Wibisono [22] showed that this isoperimetry condition is sufficient for rapid convergence of Langevin diffusion in Rényi divergence. Under LSI, they provide the following useful lower bound on I_α/E_α for an arbitrary pair of distributions.

Lemma 3 ([22] c-LSI in terms of Rényi Divergence). *Suppose $\Theta_t, \Theta'_t \in \mathbb{R}^d$ are random variables such that the density ratio between distributions of Θ_t and Θ'_t lies in $\mathcal{F}_{\Theta'_t}$. Then for any $\alpha \geq 1$,*

$$R_\alpha(\Theta_t \| \Theta'_t) + \alpha(\alpha - 1) \frac{\partial R_\alpha(\Theta_t \| \Theta'_t)}{\partial \alpha} \leq \frac{\alpha^2}{2c} \frac{I_\alpha(\Theta_t \| \Theta'_t)}{E_\alpha(\Theta_t \| \Theta'_t)}, \quad (12)$$

if and only if distribution of Θ'_t satisfies c-LSI.

Note that $\frac{\partial R_\alpha(\Theta_t \| \Theta'_t)}{\partial \alpha}$ is always positive, as $R_\alpha(\Theta_t \| \Theta'_t)$ monotonically increases with $\alpha > 1$ [17]. This lemma shows that $I_\alpha(\Theta_t \| \Theta'_t)/E_\alpha(\Theta_t \| \Theta'_t)$ grows monotonically with the Rényi privacy loss $R_\alpha(\Theta_t \| \Theta'_t)$. By Lemma 2, this implies a throttling privacy loss rate as privacy loss accumulates. Combining Lemma 2 and Lemma 3, we therefore model the **dynamics for Rényi privacy loss under c-LSI** with the following PDE, which describes the relation between privacy loss, its changes over time, and its change over Rényi parameter α . For brevity, let $R(\alpha, t)$ represent $R_\alpha(\Theta_t \| \Theta'_t)$.

$$\frac{\partial R(\alpha, t)}{\partial t} \leq \frac{1}{\gamma} \frac{\alpha S_v^2}{4\sigma^2} - 2(1 - \gamma)\sigma^2 c \left[\frac{R(\alpha, t)}{\alpha} + (\alpha - 1) \frac{\partial R(\alpha, t)}{\partial \alpha} \right] \quad (13)$$

The initial privacy loss $R(\alpha, 0) = 0$, as $\Theta_0 = \Theta'_0$. The solution for this PDE increases with time $t \geq 0$, and models the erosion of Rényi privacy loss in coupled tracing diffusions Θ_t and Θ'_t .

3.3 Privacy guarantee for Noisy GD

We now use the privacy dynamics (13) of coupled tracing diffusions to analyze the privacy dynamics for noisy GD. We first bound the difference between the underlying vector fields V_t and V'_t for coupled tracing diffusions for noisy GD on neighboring datasets D and D' .

Lemma 4. *Let $\ell(\theta; \mathbf{x})$ be a loss function on closed convex set \mathcal{C} , with a finite total gradient sensitivity S_g . Let $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ be the coupled tracing diffusions for noisy GD on neighboring datasets $D, D' \in \mathcal{X}^n$, under loss $\ell(\theta; \mathbf{x})$ and noise variance σ^2 . Then the difference between underlying vector fields V_t and V'_t for coupled tracing diffusions is bounded by*

$$\max_{\theta \in \mathbb{R}^d} \|V_t(\theta) - V'_t(\theta)\|_2 \leq \frac{S_g}{n}, \quad (14)$$

where $V_t(\theta)$ and $V'_t(\theta)$ are time-dependent vector fields on \mathbb{R}^d , defined in Lemma 1.

We then substitute S_v in PDE (13) with S_g/n , and compute the following PDE modelling Rényi privacy loss dynamics of tracing diffusion at $\eta k < t < \eta(k+1)$, under c -LSI condition.

$$\frac{\partial R(\alpha, t)}{\partial t} \leq \frac{1}{\gamma} \frac{\alpha S_g^2}{4\sigma^2 n^2} - 2(1-\gamma)\sigma^2 c \left[\frac{R(\alpha, t)}{\alpha} + (\alpha-1) \frac{\partial R(\alpha, t)}{\partial \alpha} \right] \quad (15)$$

We solve this PDE under $\gamma = \frac{1}{2}$ for each time piece, and combine multiple pieces by seeing projection as privacy-preserving post-processing step. We derive the RDP guarantee for the Noisy GD algorithm.

Theorem 2 (RDP for noisy GD under c -LSI). *Let $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ be the tracing diffusion for $\mathcal{A}_{\text{Noisy-GD}}$ on neighboring datasets D and D' , under noise variance σ^2 and loss function $\ell(\theta; \mathbf{x})$. Let $\ell(\theta; \mathbf{x})$ be a loss function on closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$, with a finite total gradient sensitivity S_g . If for any neighboring datasets D and D' , the corresponding coupled tracing diffusions Θ_t and Θ'_t satisfy c -LSI throughout $0 \leq t \leq \eta K$, then $\mathcal{A}_{\text{Noisy-GD}}$ satisfies (α, ε) Rényi Differential Privacy for*

$$\varepsilon = \frac{\alpha S_g^2}{2c\sigma^4 n^2} (1 - e^{-\sigma^2 c \eta K}). \quad (16)$$

This theorem offers a strong converging privacy guarantee, on the condition that c -LSI is satisfied throughout the Noisy GD process. We then analyze the LSI constant c for given Noisy GD process.

Isoperimetry constants for noisy GD When the loss function is strongly convex and smooth, we prove that tracing diffusion of noisy GD satisfies LSI. This is because the gradient descent update is Lipschitz under smooth loss, and the Gaussian noise preserves LSI, as discussed in Appendix D.3.

Lemma 5 (LSI for noisy GD). *If loss function $\ell(\theta; \mathbf{x})$ is λ -strongly convex and β -smooth over a closed convex set \mathcal{C} , the step-size is $\eta < \frac{1}{\beta}$, and initial distribution is $\Theta_0 \sim \Pi_{\mathcal{C}}(\mathcal{N}(0, \frac{2\sigma^2}{\lambda} \mathbb{I}_d))$, then the coupled tracing diffusion processes $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ for noisy GD on any neighboring datasets D and D' satisfy c -LSI for any $t \geq 0$ with $c = \frac{\lambda}{2\sigma^2}$.*

Using the LSI constant proved by this lemma, we immediately prove the following RDP bound for noisy GD on Lipschitz smooth strongly convex loss, as a corollary of Theorem 2.

Corollary 1 (Privacy Guarantee for noisy GD). *Let $\ell(\theta; \mathbf{x})$ be a λ -strongly convex, and β -smooth loss function on closed convex set \mathcal{C} , with a finite total gradient sensitivity S_g , then the noisy gradient descent algorithm (Algorithm 1) with start parameter $\theta_0 \sim \Pi_{\mathcal{C}}(\mathcal{N}(0, \frac{2\sigma^2}{\lambda} \mathbb{I}_d))$, and step-size $\eta < \frac{1}{\beta}$, satisfies (α, ε) Rényi Differential Privacy with*

$$\varepsilon = \frac{\alpha S_g^2}{\lambda \sigma^2 n^2} (1 - e^{-\lambda \eta K/2}).$$

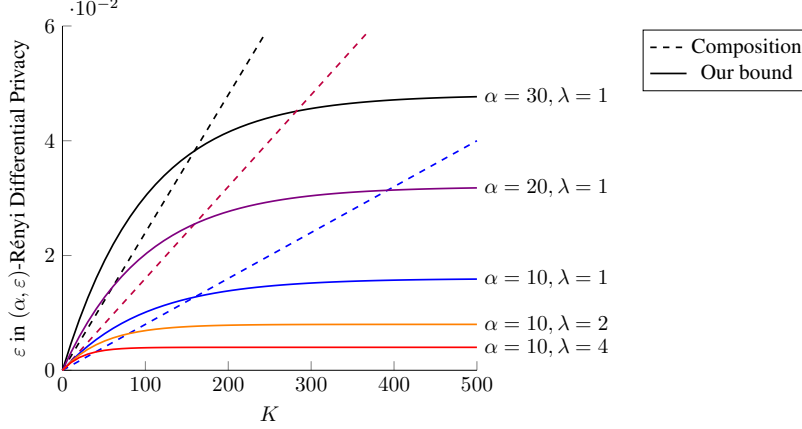


Figure 1: Rényi privacy loss of noisy GD over K iterations, quantified using our DP Dynamics Analysis. We show ε in the (α, ε) -RDP guarantee derived by Corollary 1 (bold lines), and the Baseline composition analysis (dashed lines). We evaluate under the following setting: RDP order $\alpha \in \{10, 20, 30\}$; λ -strongly convex loss function with $\lambda \in \{1, 2, 4\}$; β -smooth loss function with $\beta = 4$; finite ℓ_2 -sensitivity S_g for total gradient with $S_g = 4$; size of the data set $n = 5000$; step-size $\eta = 0.02$; noise standard deviation $\sigma = 0.02$. The expressions for computing the privacy loss are: our analysis: $\varepsilon = \frac{\alpha S_g^2}{\lambda \sigma^2 n^2} \cdot (1 - e^{-\lambda \eta K/2})$; and Baseline composition-based analysis (derived by moment accountant [1] with details in Appendix E): $\varepsilon = \frac{\alpha S_g^2}{4n^2 \sigma^2} \cdot \eta K$.

This privacy bound has quadratic dependence on the total gradient sensitivity S_g , which is upper bounded by $S_g \leq 2L$ for L -Lipschitz loss functions. The smoothness condition β restricts the step-size and ensures Lipschitz gradient mapping, thus facilitating LSI by Lemma 5. Figure 1 demonstrates how this RDP guarantee for noisy GD converges with the number of iterations K . Through y-axis, we show the ε guaranteed for noisy GD under various Rényi divergence orders c and strong convexity constant λ . The RDP order α linearly scales the asymptotic guarantee, but does not affect the convergence rate of RDP guarantee. However, the strong convexity parameter λ positively affects the asymptotic guarantee as well as the convergence rate; the larger the strong convexity parameter λ is, the stronger the asymptotic RDP guarantee and the faster the convergence.

4 Tightness analysis: a lower bound on privacy loss of noisy GD

Differential privacy guarantees reflect a *bound* on privacy loss on an algorithm; thus, it is very crucial to also have an analysis of their tightness (i.e., how close they are to the exact privacy loss). We prove that our RDP guarantee in Theorem 2 is tight. To this end, we construct an instance of the ERM optimization problem, for which we show that the Rényi privacy loss of the noisy GD algorithm grows at an order matching our guarantee in Theorem 2.

It is very challenging to lower bound the the exact Rényi privacy loss $R_\alpha(\Theta_{K\eta} \parallel \Theta'_{K\eta})$ in general. This might require having an explicit expression for the probability distribution over the last-iterate parameters θ_k . Computing a close-form expression is, however, feasible when the loss gradients are linear. This is due to the fact that, after a sequence of linear transformations and Gaussian noise additions, the parameters follow a Gaussian distribution. Therefore, we construct such an ERM objective, compute the exact privacy loss, and prove the following lower bound.

Theorem 3 (Lower bound on RDP of $\mathcal{A}_{\text{Noisy-GD}}$). *There exist two neighboring datasets $D, D' \in \mathcal{X}^n$, a start parameter θ_0 , and a smooth loss function $\ell(\theta; \mathbf{x})$ on unconstrained convex set $\mathcal{C} = \mathbb{R}^d$, with a finite total gradient sensitivity S_g , such that for any step-size $\eta < 1$, noise variance $\sigma^2 > 0$, and iteration $K \in \mathbb{N}$, the privacy loss of $\mathcal{A}_{\text{Noisy-GD}}$ on D, D' is lower-bounded by*

$$R_\alpha(\Theta_{\eta K} \parallel \Theta'_{\eta K}) \geq \frac{\alpha S_g^2}{4\sigma^2 n^2} (1 - e^{-\eta K}). \quad (17)$$

We prove this lower bound using the ℓ_2 -squared norm loss as ERM objective: $\min_{\theta \in \mathbb{R}^d} \sum_{i=1}^n \frac{\|\theta - \mathbf{x}_i\|_2^2}{n}$. We assume bounded data domain s.t. the gradient has finite sensitivity. With start parameter $\theta_0 = 0^d$, the k^{th} step parameter θ_k is distributed as Gaussian with mean $\mu_k = \eta \bar{\mathbf{x}} \sum_{i=0}^{k-1} (1-\eta)^i$ and variance $\sigma_k^2 = \frac{2\eta\sigma^2}{n^2} \sum_{i=0}^{k-1} (1-\eta)^{2i}$ in each dimension, where $\bar{\mathbf{x}} = \sum_{i=1}^n \mathbf{x}_i/n$ is the empirical dataset mean. We explicitly compute the privacy loss at any step K , which is lower bounded by $\frac{\alpha S_g^2}{4\sigma^2 n^2} (1 - e^{-\eta K})$.

Meanwhile, Corollary 1 gives our RDP upper bound $\epsilon = \frac{\alpha S_g^2}{\sigma^2 n^2} (1 - e^{-\eta K})$ for this same ERM objective. This upper bound matches the lower bound at every step K , up to a small constant of 4.

Moreover, Theorem 2 facilitates a smaller RDP upper bound than Corollary 1 by bounding the LSI constant throughout Noisy GD exactly. For squared-norm loss function, Theorem 2 gives the following tighter RDP upper-bound for Noisy GD, because all intermediate Gaussian distributions satisfy c -LSI with $c = \frac{2-\eta}{2\sigma^2}$, as proved in Appendix E.

Corollary 2 (RDP guarantee of $\mathcal{A}_{\text{Noisy-GD}}$ on ℓ_2 -norm squared loss). *For any two neighboring datasets $D, D' \in \mathcal{X}^n$, start parameter θ_0 , step-size η , noise variance σ^2 , and $K \in \mathbb{N}$, if the loss function is ℓ_2 -norm squared loss $\ell(\theta; \mathbf{x}) = \frac{1}{2} \|\theta - \mathbf{x}\|_2^2$ on unconstrained convex set $\mathcal{C} = \mathbb{R}^d$, with a finite total gradient sensitivity S_g , the privacy loss of $\mathcal{A}_{\text{Noisy-GD}}$ on D, D' is upper-bounded by*

$$R_\alpha(\Theta_{\eta K} \|\Theta'_{\eta K}) \leq \frac{\alpha S_g^2}{(2-\eta)\sigma^2 n^2} (1 - e^{-\frac{2-\eta}{2}\eta K}). \quad (18)$$

This RDP guarantee converges fast to $O(\frac{\alpha S_g^2}{\sigma^2 n^2})$, which matches the lower bound at every step K , up to a constant of $\frac{4}{2-\eta} \approx 2$. This immediately shows tightness of our converging RDP guarantee *throughout* the training process, for a converging noisy GD algorithm. A different approach is to completely ignore the dynamics of differential privacy, and instead analyze privacy *only at the convergence time* (or when the algorithm is near convergence). Wang et al. [25], Minami et al. [16] show that sampling from the Gibbs posterior distribution $\nu(\theta) \propto e^{-\mathcal{L}_D(\theta)/\sigma^2}$ for bounded \mathcal{L}_D satisfies differential privacy. However, sampling exactly from the Gibbs distribution is difficult [4]. Thus, Minami et al. [16], Ganesh and Talwar [11] extend the DP guarantees of Gibbs posterior distribution to gradient-descent based samplers such as Unadjusted Langevin Algorithm (ULA) that can sample from distributions arbitrarily close to Gibbs distribution after a sufficient number of iterations K with extremely small step-size η . Minami et al. [16] compute the distance to convergence in total variation, and Ganesh and Talwar [11] improve the prior bound by measuring the distance in Rényi divergence (building on the rapid convergence results of Vempala and Wibisono [22]). The latter results in a better gradient complexity $\Omega(nd)$, which however still grows with model dimension d . In comparison, our DP guarantees are unaffected by parameter dimension d , which in practice can be much larger than the dataset size n .

In contrast, composition-based privacy bound grows linearly as training proceeds, as shown in Figure 1. When the number of iterations K is small, however, composition-based bound grows at the same rate with the lower bound, as discussed in Appendix E. Therefore, to conclude whether our RDP guarantee is superior to composition-based bound, we need to understand the number of iterations noisy GD needs, to achieve optimal utility. We discuss this in the following section.

5 Utility analysis for noisy gradient descent

The randomness, required for satisfying differential privacy, can adversely affect the utility of the trained model. The standard way to measure the utility of a randomized ERM algorithm (for example, $\mathcal{A}_{\text{Noisy-GD}}$) is to quantify its worst case *excess empirical risk*, which is defined as

$$\max_{D \in \mathcal{X}^n} \mathbb{E}[\mathcal{L}_D(\theta) - \mathcal{L}_D(\theta^*)], \quad (19)$$

where θ is the output of the randomized algorithm $\mathcal{A}_{\text{Noisy-GD}}$ on D , θ^* is the solution to the standard (no privacy) ERM (3), and the expectation is computed over the randomness of the algorithm.

We provide the *optimal* excess empirical risk (utility) of noisy GD algorithm under (α, ϵ') -RDP constraint. The notion of *optimality* for utility is defined as the smallest upper-bound for excess

Table 1: Utility comparison with the prior (ϵ, δ) -DP ERM algorithms. We assume 1-Lipschitz, β -smooth and λ -strongly convex loss. Size of input dataset is n , and dimension of parameter vector θ is d . For objective perturbation, we assume $\epsilon \geq \frac{\beta}{2\lambda}$, and loss is twice differentiable. For our result, we assume $\epsilon \leq 2 \log(1/\delta)$. The lower bound is $\Omega(\min\{1, \frac{d}{\epsilon^2 n^2}\})$ [3]. We ignore numerical constants and multiplicative dependence on $\log(1/\delta)$.

	Method	Utility Upper Bound	Gradient complexity
Bassily et al. [3]	Noisy SGD	$O(\frac{d \log^2(n)}{\lambda n^2 \epsilon^2})$	n^2
Wang et al. [23]	DP-SVRG	$O(\frac{d \log(n)}{\lambda n^2 \epsilon^2})$	$O\left(\left(n + \frac{\beta}{\lambda}\right) \log\left(\frac{\lambda n^2 \epsilon^2}{d}\right)\right)$
Zhang et al. [26]	Output Perturbation	$O(\frac{\beta d}{\lambda^2 n^2 \epsilon^2})$	$O\left(\frac{\beta}{\lambda} n \log\left(\frac{n^2 \epsilon^2}{d}\right)\right)$
Kifer et al. [14]	Objective Perturbation	$O(\frac{d}{\lambda n^2 \epsilon^2})$	NA
This Paper	Noisy GD	$O(\frac{\beta d}{\lambda^2 n^2 \epsilon^2})$	$O\left(\frac{\beta^2}{\lambda^2} n \log\left(\frac{n^2 \epsilon^2}{d}\right)\right)$

empirical risk that can be guaranteed under (α, ϵ') -RDP constraint by tuning the algorithm's hyper-parameters (such as the noise variance σ^2 and the number of iterations K). We focus here on smooth and strongly convex loss functions with a finite total gradient sensitivity.

Lemma 6 (Excess empirical risk for smooth and strongly convex loss). *For L -Lipschitz, λ -strongly convex and β -smooth loss function $\ell(\theta; \mathbf{x})$ over a closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$, step-size $\eta \leq \frac{\lambda}{2\beta^2}$, and start parameter $\theta_0 \sim \Pi_{\mathcal{C}}(\mathcal{N}(0, \frac{2\sigma^2}{\lambda} \mathbb{I}_d))$, the excess empirical risk of Algorithm 1 is bounded by*

$$\mathbb{E}[\mathcal{L}_D(\theta_K) - \mathcal{L}_D(\theta^*)] \leq \frac{2\beta L^2}{\lambda^2} e^{-\lambda\eta K} + \frac{2\beta d\sigma^2}{\lambda}, \quad (20)$$

where θ^* is the minimizer of $\mathcal{L}_D(\theta)$ in the relative interior of convex set \mathcal{C} , and d is the dimension of parameter.

This lemma shows decreasing excess empirical risk for noisy GD algorithm under Lipschitz smooth strongly convex loss function as the number of iterations K increases. The utility is determined by K and the noise variance σ^2 , which are constrained under (α, ϵ') -RDP. Using our tight RDP guarantee in Corollary 1, we prove optimal utility for noisy GD.

Theorem 4 (Upper bound for (α, ϵ') -RDP and (ϵ, δ) -DP Noisy GD). *For Lipschitz smooth strongly convex loss function $\ell(\theta; \mathbf{x})$ on a bounded closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$, and dataset $D \in \mathcal{X}^n$ of size n , if the step-size $\eta = \frac{\lambda}{2\beta^2}$ and the initial parameter $\theta_0 \sim \Pi_{\mathcal{C}}(\mathcal{N}(0, \frac{2\sigma^2}{\lambda} \mathbb{I}_d))$, then the noisy GD Algorithm 1 is (α, ϵ') -Rényi differentially private, where $\alpha > 1$ and $\epsilon' > 0$, and satisfies*

$$\mathbb{E}[\mathcal{L}_D(\theta_{K^*}) - \mathcal{L}_D(\theta^*)] = O\left(\frac{\alpha\beta dL^2}{\epsilon' \lambda^2 n^2}\right), \quad (21)$$

by setting noise variance $\sigma^2 = \frac{4\alpha L^2}{\lambda \epsilon' n^2}$, and number of updates $K^* = \frac{2\beta^2}{\lambda^2} \log\left(\frac{n^2 \epsilon'}{\alpha d}\right)$.

Equivalently, for $\epsilon \leq 2 \log(1/\delta)$ and $\delta > 0$, Algorithm 1 is (ϵ, δ) -differentially private, and satisfies

$$\mathbb{E}[\mathcal{L}_D(\theta_{K^*}) - \mathcal{L}_D(\theta^*)] = O\left(\frac{\beta dL^2 \log(1/\delta)}{\epsilon^2 \lambda^2 n^2}\right), \quad (22)$$

by setting noise variance $\sigma^2 = \frac{8L^2(\epsilon+2 \log(1/\delta))}{\lambda \epsilon^2 n^2}$, and number of updates $K^* = \frac{2\beta^2}{\lambda^2} \log\left(\frac{n^2 \epsilon^2}{4 \log(1/\delta) d}\right)$.

Our algorithm achieves this utility guarantee with $O\left(\frac{\beta^2}{\lambda^2} n \log\left(\frac{\epsilon^2 n^2}{d}\right)\right)$ gradient computations of $\nabla \ell(\theta; \mathbf{x})$, which is faster than noisy SGD algorithm [3] with a factor of n . However, we additionally assume smoothness for the loss function. Our gradient complexity also matches that of other efficient gradient perturbation and output perturbation methods [23, 26], as shown in Table 1.

This utility matches the following theoretical lower bound in Bassily et al. [3] for the best attainable utility of (ϵ, δ) -differentially private algorithms on Lipschitz smooth strongly convex loss functions.

Theorem 5 ([3] Lower bound for (ε, δ) -DP algorithms). *Let $n, d \in \mathbb{N}$, $\varepsilon > 0$, and $\delta = o(\frac{1}{n})$. For every (ε, δ) -differentially private algorithm \mathcal{A} (whose output is denoted by θ^{priv}), there is a dataset $D \in \mathcal{X}^n$ such that, with probability at least $1/3$ (over the algorithm random coins), we must have*

$$\mathcal{L}_D(\theta^{priv}) - \mathcal{L}_D(\theta^*) = \Omega\left(\min\left\{1, \frac{d}{\varepsilon^2 n^2}\right\}\right), \quad (23)$$

where θ^* minimizes a constructed 1-Lipschitz, 1-strongly convex objective $\mathcal{L}_D(\theta)$ over convex set \mathcal{C} .

Our utility matches this lower bound upto the constant factor $\log(1/\delta)$, when assuming $\frac{\beta}{\lambda^2} = O(1)$. This improves upon the previous gradient perturbation methods [3, 24] by a factor of $\log(n)$, and matches the utility of previously know optimal ERM algorithm for Lipschitz smooth strongly convex loss functions, such as objective perturbation [6, 14] and output perturbation [26].

Utility gain from tight privacy guarantee As shown in Table 1, our utility guarantee for noisy GD is logarithmically better than that for noisy SGD in Bassily et al. [3], although the two algorithms are extremely similar. This is because we use our tight RDP guarantee, while Bassily et al. [3] use a composition-based privacy bound. More specifically, noisy SGD needs n^2 iterations to achieve the optimal utility, as shown in Table 1. This number of iterations is large enough for the composition-based privacy bound to grow above our RDP guarantee, thus leaving room for improving privacy utility trade-off, as we further discuss in Appendix F. This concludes that our tight privacy guarantee enables providing a superior privacy-utility trade-off, for Lipschitz, strongly convex, and smooth loss functions.

Our algorithm also has significantly smaller gradient complexity than noisy SGD [3], for strongly convex loss functions, by a factor of $n/\log n$. We use a (moderately large) constant step-size, thus achieving fast convergence to optimal utility. However, noisy SGD [3] uses a decreasing step-size, thus requiring more iterations to reach optimal utility.

6 Conclusions

We have developed a novel theoretical framework for analyzing the dynamics of privacy loss for noisy gradient descent algorithms. Our theoretical results show that by hiding the internal state of the training algorithm (over many iterations over the whole data), we can tightly analyze the rate of information leakage throughout training, and derive a bound that is significantly tighter than that of composition-based approaches.

Future Work. Our main result is a tight privacy guarantee for Noisy GD on smooth and strongly convex loss functions. The assumptions are very similar to that of the prior work on privacy amplification by iteration [8], and have obvious advantages in enabling the tightness and utility analysis. However, the remaining open challenge is to extend this analysis to non-smooth and non-convex loss functions, and stochastic gradient updates, which are used notably in deep learning.

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References

- [1] Martin Abadi, Andy Chu, Ian Goodfellow, H Brendan McMahan, Ilya Mironov, Kunal Talwar, and Li Zhang. Deep learning with differential privacy. In *Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security*, pages 308–318, 2016.

- [2] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348. Springer Science & Business Media, 2013.
- [3] Raef Bassily, Adam Smith, and Abhradeep Thakurta. Private empirical risk minimization: Efficient algorithms and tight error bounds. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 464–473. IEEE, 2014.
- [4] William M Bolstad. *Understanding computational Bayesian statistics*, volume 644. John Wiley & Sons, 2009.
- [5] Nicholas Carlini, Florian Tramer, Eric Wallace, Matthew Jagielski, Ariel Herbert-Voss, Katherine Lee, Adam Roberts, Tom Brown, Dawn Song, Ulfar Erlingsson, et al. Extracting training data from large language models. *arXiv preprint arXiv:2012.07805*, 2020.
- [6] Kamalika Chaudhuri, Claire Monteleoni, and Anand D Sarwate. Differentially private empirical risk minimization. *Journal of Machine Learning Research*, 12(Mar):1069–1109, 2011.
- [7] Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. *Foundations and Trends® in Theoretical Computer Science*, 9(3–4):211–407, 2014.
- [8] Vitaly Feldman, Ilya Mironov, Kunal Talwar, and Abhradeep Thakurta. Privacy amplification by iteration. In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 521–532. IEEE, 2018.
- [9] Vitaly Feldman, Tomer Koren, and Kunal Talwar. Private stochastic convex optimization: optimal rates in linear time. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 439–449, 2020.
- [10] Adriaan Daniël Fokker. Die mittlere energie rotierender elektrischer dipole im strahlungsfeld. *Annalen der Physik*, 348(5):810–820, 1914.
- [11] Arun Ganesh and Kunal Talwar. Faster differentially private samplers via Rényi divergence analysis of discretized Langevin MCMC. *Advances in Neural Information Processing Systems*, 33, 2020.
- [12] Leonard Gross. Logarithmic sobolev inequalities. *American Journal of Mathematics*, 97(4):1061–1083, 1975.
- [13] Matthew Jagielski, Jonathan Ullman, and Alina Oprea. Auditing differentially private machine learning: How private is private sgd? *Advances in Neural Information Processing Systems*, 33, 2020.
- [14] Daniel Kifer, Adam Smith, and Abhradeep Thakurta. Private convex empirical risk minimization and high-dimensional regression. In *Conference on Learning Theory*, pages 25–1. JMLR Workshop and Conference Proceedings, 2012.
- [15] Michel Ledoux. *The concentration of measure phenomenon*. Number 89. American Mathematical Soc., 2001.
- [16] Kentaro Minami, Hitomi Arai, Issei Sato, and Hiroshi Nakagawa. Differential privacy without sensitivity. In *Advances in Neural Information Processing Systems*, pages 956–964, 2016.
- [17] Ilya Mironov. Rényi differential privacy. In *2017 IEEE 30th Computer Security Foundations Symposium (CSF)*, pages 263–275. IEEE, 2017.
- [18] Milad Nasr, Reza Shokri, and Amir Houmansadr. Comprehensive privacy analysis of deep learning: Passive and active white-box inference attacks against centralized and federated learning. In *2019 IEEE symposium on security and privacy (SP)*, pages 739–753. IEEE, 2019.
- [19] Milad Nasr, Shuang Song, Abhradeep Thakurta, Nicolas Papernot, and Nicholas Carlini. Adversary instantiation: Lower bounds for differentially private machine learning. *arXiv preprint arXiv:2101.04535*, 2021.
- [20] Issei Sato and Hiroshi Nakagawa. Approximation analysis of stochastic gradient Langevin dynamics by using fokker-planck equation and ito process. In *International Conference on Machine Learning*, pages 982–990, 2014.
- [21] Reza Shokri, Marco Stronati, Congzheng Song, and Vitaly Shmatikov. Membership inference attacks against machine learning models. In *2017 IEEE Symposium on Security and Privacy (SP)*, pages 3–18. IEEE, 2017.

- [22] Santosh Vempala and Andre Wibisono. Rapid convergence of the unadjusted Langevin algorithm: Isoperimetry suffices. In *Advances in Neural Information Processing Systems*, pages 8094–8106, 2019.
- [23] Di Wang, Minwei Ye, and Jinhui Xu. Differentially private empirical risk minimization revisited: Faster and more general. In *Advances in Neural Information Processing Systems*, pages 2722–2731, 2017.
- [24] Di Wang, Minwei Ye, and Jinhui Xu. Differentially private empirical risk minimization revisited: Faster and more general. *arXiv preprint arXiv:1802.05251*, 2018.
- [25] Yu-Xiang Wang, Stephen Fienberg, and Alex Smola. Privacy for free: Posterior sampling and stochastic gradient monte carlo. In *International Conference on Machine Learning*, pages 2493–2502, 2015.
- [26] Jiaqi Zhang, Kai Zheng, Wenlong Mou, and Liwei Wang. Efficient private erm for smooth objectives. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence*, pages 3922–3928, 2017.

Appendix

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A Table of Notations

Table 2: Symbol reference

Symbol	Meaning
d	Dimension of model parameters.
\mathbb{R}^d	Unconstrained model parameter space of dimension d .
\mathcal{C}	A closed convex Model parameter set $\mathcal{C} \subseteq \mathbb{R}^d$ for convex optimization.
$\Pi_{\mathcal{C}}(\theta)$	Projection of θ to \mathcal{C} .
$\mathcal{X}, \mathcal{X}^n$	Data universe and Domain of all datasets of size n .
n	Dataset size.
D, D'	Neighbouring Dataset of size n .
\mathbf{x}_i	i -th data point in dataset D .
$\ell(\theta; \mathbf{x})$	Risk of parameter θ w.r.t data point \mathbf{x} .
$\mathcal{L}_D(\theta)$	Empirical risk optimization objective.
$U_1(\theta)$	Average between gradients on neighbouring datasets D and D' .
$U_2(\theta)$	Half of difference between gradients on neighbouring datasets D and D' .
$g(\theta; D)$	Sum of risk gradients at θ for all data points in D .
V_t, V'_t	Time-variable vector fields on \mathbb{R}^d .
S_g	ℓ_2 -sensitivity of total loss gradient $g(\theta; D)$
S_v	maximum ℓ_2 distance between V_t and V'_t for all $t > 0$.
θ^*	Parameter minimizing the empirical risk $\mathcal{L}_D(\theta)$.
\mathcal{L}	Potential function for Langevin diffusion.
\mathbf{W}_t	Standard Brownian motion aka. Wiener process.
α	Rényi differential privacy order.
δ	Probability of uncontrolled breach in standard DP.
ε	Rényi or standard DP privacy parameter.
\mathcal{A}	Randomized algorithm.
ν, ν'	Two probability measures.
p, p'	Two Probability densities over parameter space \mathbb{R}^d .
Θ, Θ'	Two random variables distributed as p, p' respectively.
σ^2	Noise variance in noisy GD and Langevin diffusion.
\mathbb{I}_d	d -dimensional identity matrix.
$\mathcal{N}(0, \mathbb{I}_d)$	Standard gaussian distribution with dimension d .
Z, Z_1, Z_2, \dots	Random variables taken from $\mathcal{N}(0, \mathbb{I}_d)$.
η	Step size of updates in noisy GD.
λ	Strong convexity parameter of risk function.
β	Smoothness parameter of risk function.
B	Bound on range of risk function.
L	Lipschitzness parameter of risk function.
K, k	Number of update steps and intermediate step index in noisy GD.
θ_k, θ'_k	Parameter at step k of noisy GD on $\mathcal{D}, \mathcal{D}'$.
T, t	Termination time and intermediate time stamp for diffusion.
Θ_t, Θ'_t	Model parameter random variable at time t of diffusion on $\mathcal{D}, \mathcal{D}'$.
p_t, p'_t	Probability densities or random variables Θ_t, Θ'_t
p_{t_1, t_2}	Joint density between diffusion random variables $(\Theta_{t_1}, \Theta_{t_2})$.
p'_{t_1, t_2}	Joint density between diffusion random variables $(\Theta'_{t_1}, \Theta'_{t_2})$.
$p_{t_1 t_2}(\theta \theta_{t_2})$	Conditional density for Θ_{t_1} given $\Theta_{t_2} = \theta_{t_2}$.
$p'_{t_1 t_2}(\theta \theta_{t_2})$	Conditional density for Θ'_{t_1} given $\Theta'_{t_2} = \theta_{t_2}$.
$R_\alpha(\Theta_t \Theta'_t)$	Rényi divergence of distribution of Θ_t w.r.t Θ'_t .
$E_\alpha(\Theta_t \Theta'_t)$	α^{th} moment of likelihood ratio r.v. between Θ_t, Θ'_t .
$I_\alpha(\Theta_t \Theta'_t)$	Rényi Information of distribution of Θ_t w.r.t Θ'_t .
c	Constant in Log sobolev inequality.
\ll	Absolute continuity with respect to measure.

B Preliminaries

B.1 Divergence measures

A measure ν is said to be absolutely continuous with respect to another measure ν' on same space (denoted as $\nu \ll \nu'$) if for all measurable set S , $\nu(S) = 0$ whenever $\nu'(S) = 0$.

Definition B.1 (α -Rényi Divergence). For $\alpha > 1$, and any two measures ν, ν' with $\nu \ll \nu'$, the α -Rényi Divergence $R_\alpha(\cdot \| \cdot)$ of ν with respect to ν' is defined as

$$R_\alpha(\nu \| \nu') = \frac{1}{\alpha - 1} \log E_\alpha(\nu \| \nu'), \quad (24)$$

where $E_\alpha(\nu \| \nu')$ is defined as:

$$E_\alpha(\nu \| \nu') = \int \left(\frac{d\nu}{d\nu'} \right)^\alpha d\nu', \quad (25)$$

Additionally, if ν and ν' are absolutely continuous with Lebesgue measures on \mathbb{R}^d (i.e. they are continuous distributions on \mathbb{R}^d) with densities p and p' respectively, $E_\alpha(\nu \| \nu')$ is same as

$$E_\alpha(\nu \| \nu') = \mathbb{E}_{\theta \sim p'} \left[\frac{p(\theta)^\alpha}{p'(\theta)^\alpha} \right]. \quad (26)$$

As an example, the α -Rényi divergence between two Gaussian distributions centered at $\mu, \mu' \in \mathbb{R}^d$, with covariance matrix $\sigma^2 \mathbb{I}_d$ is $\frac{\alpha \|\mu - \mu'\|_2^2}{2\sigma^2}$ [17, Proposition 7].

Definition B.2 (Rényi information [22]). Let $1 < \alpha < \infty$. For any two measures ν, ν' with $\nu \ll \nu'$, if the Radon-Nikodym derivative $\frac{d\nu}{d\nu'}$ is differentiable, the α -Rényi Information $I_\alpha(\cdot \| \cdot)$ of ν with respect to ν' is

$$I_\alpha(\nu \| \nu') = \int \left(\frac{d\nu}{d\nu'} \right)^\alpha \left\| \nabla \log \frac{d\nu}{d\nu'} \right\|_2^2 d\nu'. \quad (27)$$

Additionally, if ν and ν' are absolutely continuous with Lebesgue measures (i.e. they are continuous distributions on \mathbb{R}^d) with densities p and p' respectively, $I_\alpha(\nu \| \nu')$ is same as

$$I_\alpha(\nu \| \nu') = \frac{4}{\alpha^2} \mathbb{E}_{\theta \sim p'} \left[\left\| \nabla \frac{p(\theta)^{\frac{\alpha}{2}}}{p'(\theta)^{\frac{\alpha}{2}}} \right\|_2^2 \right] = \mathbb{E}_{\theta \sim p'} \left[\frac{p(\theta)^{\alpha-2}}{p'(\theta)^{\alpha-2}} \left\| \nabla \frac{p(\theta)}{p'(\theta)} \right\|_2^2 \right]. \quad (28)$$

B.2 Differential privacy

Let \mathcal{X} be a data universe. Let a dataset be a vector of n records from \mathcal{X} : $D = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \in \mathcal{X}^n$.

Definition B.3 (Neighboring datasets). Two datasets D and D' are neighboring, denoted by $D \sim D'$, if $|D| = |D'|$, and they differ in exactly one data record, i.e., $|D \oplus D'| = 2$.

Definition B.4 (Differential privacy [7]). A randomized algorithm $\mathcal{A} : \mathcal{X}^n \rightarrow \mathbb{R}^d$ satisfies (ϵ, δ) -differential privacy (DP) if for any two neighboring datasets $D, D' \in \mathcal{X}^n$, and for all sets $S \in \mathbb{R}^d$,

$$Pr[\mathcal{A}(D) \in S] \leq e^\epsilon Pr[\mathcal{A}(D') \in S] + \delta. \quad (29)$$

Definition B.5 (Rényi differential privacy [17]). Let $\alpha > 1$. A randomized algorithm $\mathcal{A} : \mathcal{X}^n \rightarrow \mathbb{R}^d$ satisfies (α, ϵ) -Rényi Differential Privacy (RDP), if for any two neighboring datasets $D, D' \in \mathcal{X}^n$:

$$R_\alpha(\mathcal{A}(D) \| \mathcal{A}(D')) \leq \epsilon. \quad (30)$$

In this paper, we mainly use Rényi differential privacy notion to analyze the privacy loss of algorithms. We refer to $R_\alpha(\mathcal{A}(D) \| \mathcal{A}(D'))$ as the Rényi privacy loss of algorithm \mathcal{A} on datasets D, D' .

Theorem 6 (RDP composition theorem [17, Proposition 1]). *Let $\mathcal{A}_1 : \mathcal{X}^n \rightarrow \mathbb{R}^d$ and $\mathcal{A}_2 : \mathbb{R}^d \times \mathcal{X}^n \rightarrow \mathbb{R}^d$ be two randomized algorithms that satisfy (α, ε_1) and (α, ε_2) -RDP, respectively. The composed algorithm defined as $\mathcal{A}(D) = (\mathcal{A}_1(D), \mathcal{A}_2(D))$ satisfies $(\alpha, \varepsilon_1 + \varepsilon_2)$ -Rényi DP.*

An RDP guarantee can be converted to a DP guarantee as per the following theorem.

Theorem 7 (DP Conversion [17, Proposition 3]). *If a randomized algorithm $\mathcal{A} : \mathcal{X}^n \rightarrow \mathbb{R}^d$ satisfies (α, ε) -RDP, then it also satisfies the standard $(\varepsilon + \frac{\log 1/\delta}{\alpha-1}, \delta)$ -DP guarantee for any $0 < \delta < 1$.*

B.3 Langevin diffusion

We focus on the Langevin diffusion process in \mathbb{R}^d with noise variance σ^2 , described by the following stochastic differential equation (SDE).

$$d\Theta_t = -\nabla\mathcal{L}(\Theta_t)dt + \sqrt{2\sigma^2}d\mathbf{W}_t, \quad (31)$$

where $d\mathbf{W}_t = \mathbf{W}_{t+dt} - \mathbf{W}_t \sim \sqrt{dt}\mathcal{N}(0, \mathbb{I}_d)$ characterizes the d -dimensional Wiener process.

The joint effect of this drag force (i.e. $-\nabla\mathcal{L}$) and Brownian fluctuations on the probability density p_t of position random variable Θ_t is characterized through the Fokker-Planck equation [10],

$$\frac{\partial p_t(\theta)}{\partial t} = \nabla \cdot (p_t(\theta)\nabla\mathcal{L}(\theta)) + \sigma^2\Delta p_t(\theta), \quad (32)$$

which describes the rate of change in probability density at any position $\theta \in \mathbb{R}^d$. It's important to point out that Fokker-Planck equation isn't a property of Langevin diffusion, but rather a general equation quantifying the distributional change under *any* drag force in presence of Brownian fluctuations.

Under mild regularity conditions on the potential $\mathcal{L}(\theta)$, this diffusion process has a stationary distribution ν , given by the solution to $\frac{\partial p_t(\theta)}{\partial t} = 0$, which is the following Gibbs distribution.

$$\nu(\theta) = \frac{1}{V}e^{-\mathcal{L}(\theta)/\sigma^2}, \text{ where } V = \int_{\mathbb{R}^d} e^{-\mathcal{L}(\theta)/\sigma^2} d\theta. \quad (33)$$

B.4 Loss function properties

For any data record $\mathbf{x} \in \mathcal{X}$, a loss function $\ell(\theta; \mathbf{x}) : \mathcal{C} \rightarrow \mathbb{R}$ on a closed convex set \mathcal{C} maps parameter $\theta \in \mathcal{C} \subseteq \mathbb{R}^d$ to a real value. Let $\nabla\ell(\theta; \mathbf{x})$ be its loss gradient vector with respect to θ .

Definition B.6 (Lipschitz continuity). *Function $\ell(\theta; \mathbf{x})$ is L -Lipschitz continuous if for all $\theta, \theta' \in \mathcal{C}$ and $\mathbf{x} \in \mathcal{X}$,*

$$|\ell(\theta; \mathbf{x}) - \ell(\theta'; \mathbf{x})| \leq L \|\theta - \theta'\|_2. \quad (34)$$

Definition B.7 (Smoothness). *Differentiable function $\ell(\theta; \mathbf{x})$ is β -smooth over \mathcal{C} if for all $\theta, \theta' \in \mathcal{C}$ and $\mathbf{x} \in \mathcal{X}$,*

$$\|\nabla\ell(\theta; \mathbf{x}) - \nabla\ell(\theta'; \mathbf{x})\|_2 \leq \beta \|\theta - \theta'\|_2. \quad (35)$$

Definition B.8 (Strong convexity). *Differentiable function $\ell(\theta; \mathbf{x})$ is λ -strongly convex if for all $\theta, \theta' \in \mathbb{R}^d$ and $\mathbf{x} \in \mathcal{X}$,*

$$\ell(\theta'; \mathbf{x}) \geq \ell(\theta; \mathbf{x}) + \nabla\ell(\theta; \mathbf{x})^T(\theta' - \theta) + \frac{\lambda}{2} \|\theta' - \theta\|_2^2. \quad (36)$$

Definition B.9 (Vector field sensitivity). *For two vector fields V, V' on \mathbb{R}^d , we define S_v to be the l_2 -sensitivity between them:*

$$S_v = \max_{\theta \in \mathbb{R}^d} \|V(\theta) - V'(\theta)\|_2. \quad (37)$$

Definition B.10 (Sensitivity of total gradient). *For a differentiable function $\ell(\theta; \mathbf{x})$, we define S_g to be the l_2 -sensitivity of its total gradient $g(\theta; D) = \sum_{\mathbf{x} \in D} \nabla\ell(\theta; \mathbf{x})$ on neighboring datasets $D, D' \in \mathcal{X}^n$:*

$$S_g = \max_{D \sim D'} \max_{\theta \in \mathbb{R}^d} \|g(\theta; D) - g(\theta; D')\|_2. \quad (38)$$

In Appendix C, we briefly present the basic vector calculus that we require in this paper.

C Calculus Refresher

Given a smooth function $\mathcal{L} : \Theta \rightarrow \mathbb{R}$, where $\Theta \subset \mathbb{R}^d$, its gradient $\nabla \mathcal{L} : \mathcal{X}^n \rightarrow \mathbb{R}^d$ is the vector of partial derivatives

$$\nabla \mathcal{L}(\theta) = \left(\frac{\partial \mathcal{L}(\theta)}{\partial \theta_1}, \dots, \frac{\partial \mathcal{L}(\theta)}{\partial \theta_d} \right). \quad (39)$$

Its Hessian $\nabla^2 \mathcal{L} : \Theta \rightarrow \mathbb{R}^{d \times d}$ is the matrix of second partial derivatives

$$\nabla^2 \mathcal{L}(\theta) = \left(\frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta_i \partial \theta_j} \right)_{1 \leq i, j \leq d}. \quad (40)$$

Its Laplacian $\Delta \mathcal{L} : \Theta \rightarrow \mathbb{R}$ is the trace of its Hessian $\nabla^2 \mathcal{L}$, i.e.,

$$\Delta \mathcal{L}(\theta) = \text{Tr}(\nabla^2 \mathcal{L}(\theta)). \quad (41)$$

Given a smooth vector field $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_d) : \Theta \rightarrow \mathbb{R}^d$, its divergence $\nabla \cdot \mathbf{v} : \Theta \rightarrow \mathbb{R}$ is

$$(\nabla \cdot \mathbf{v})(\theta) = \sum_{i=1}^d \frac{\partial \mathbf{v}_i(\theta)}{\partial \theta_i}. \quad (42)$$

Some identities that we would rely on:

1. Divergence of gradient is the Laplacian, i.e.,

$$\nabla \cdot \nabla \mathcal{L}(\theta) = \sum_{i=1}^d \frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta_i^2} = \Delta \mathcal{L}(\theta). \quad (43)$$

2. For any function $f : \Theta \rightarrow \mathbb{R}$ and a vector field $\mathbf{v} : \Theta \rightarrow \mathbb{R}^d$ with sufficiently fast decay to a constant at the border of Θ ,

$$\int_{\Theta} \langle \mathbf{v}(\theta), \nabla f(\theta) \rangle d\theta = - \int_{\Theta} f(\theta) (\nabla \cdot \mathbf{v})(\theta) d\theta. \quad (44)$$

3. For any two twice continuously differentiable functions $f, g : \Theta \rightarrow \mathbb{R}$, out of which at least for one the gradient decays sufficiently fast at infinity, the following also holds.

$$\int_{\Theta} f(\theta) \Delta g(\theta) d\theta = - \int_{\Theta} \langle \nabla f(\theta), \nabla g(\theta) \rangle d\theta = \int_{\Theta} g(\theta) \Delta f(\theta) d\theta. \quad (45)$$

This identity comes from the Green's first identity, which is the higher dimensional equivalent of integration by part.

4. Based on Young's inequality, for two vector fields $\mathbf{v}_1, \mathbf{v}_2 : \Theta \rightarrow \mathbb{R}^d$, and any $a, b \in \mathbb{R}$ such that $ab = 1$, the following inequality holds.

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle(\theta) \leq \frac{1}{2a} \|\mathbf{v}_1(\theta)\|_2^2 + \frac{1}{2b} \|\mathbf{v}_2(\theta)\|_2^2. \quad (46)$$

Wherever it is clear, we would drop (θ) for brevity. For example, we would represent $(\nabla \cdot \mathbf{v})(\theta)$ as only $\nabla \cdot \mathbf{v}$.

D Proofs for Section 3: Privacy analysis of noisy gradient descent

D.1 Proofs for Section 3.1: Tracing diffusion for Noisy GD

Lemma 1. For coupled tracing diffusion processes (5) in time $\eta k < t < \eta(k+1)$, the equivalent Fokker-Planck equations are

$$\begin{cases} \frac{\partial p_t(\theta)}{\partial t} = \nabla \cdot (p_t(\theta) V_t(\theta)) + \sigma^2 \Delta p_t(\theta) \\ \frac{\partial p'_t(\theta)}{\partial t} = \nabla \cdot (p'_t(\theta) V'_t(\theta)) + \sigma^2 \Delta p'_t(\theta), \end{cases} \quad (47)$$

where $V_t(\theta) = \mathbb{E}_{\theta_k \sim p_{\eta k|t}} [U_2(\theta_k)|\theta]$ and $V'_t(\theta) = \mathbb{E}_{\theta'_k \sim p'_{\eta k|t}} [-U_2(\theta_k)|\theta]$ are time-dependent vector fields on \mathbb{R}^d , and $U_2(\theta) = \frac{1}{2} [\nabla \mathcal{L}_D(\theta) - \nabla \mathcal{L}_{D'}(\theta)]$ is the difference between gradients on neighboring datasets D and D' .

Proof. We only prove $\frac{\partial p_t(\theta)}{\partial t} = \nabla \cdot (p_t(\theta)V_t(\theta)) + \sigma^2 \Delta p_t(\theta)$. The proof for the other Fokker-Planck equation is similar.

Recall that conditionals of joint distribution $p_{\eta k, t}$ is

$$p_{\eta k, t}(\theta_k, \theta) = p_{\eta k}(\theta_k)p_{t|\eta k}(\theta|\theta_k) = p_t(\theta)p_{\eta k|t}(\theta_k|\theta). \quad (48)$$

By marginalizing away θ_k in (48), and taking partial derivative w.r.t. t on both sides, we obtain the following:

$$\begin{aligned} \frac{\partial p_t(\theta)}{\partial t} &= \int_{\mathbb{R}^d} \frac{\partial p_{t|\eta k}(\theta|\theta_k)}{\partial t} p_{\eta k}(\theta_k) d\theta_k \\ &= \int_{\mathbb{R}^d} (\nabla \cdot (p_{\eta k, t}(\theta_k, \theta)U_2(\theta_k)) + \sigma^2 \Delta p_{\eta k, t}(\theta_k, \theta)) d\theta_k \quad (\text{By (8)}) \\ &= \nabla \cdot \left(p_t(\theta) \int_{\mathbb{R}^d} p_{\eta k|t}(\theta_k|\theta) U_2(\theta_k) d\theta_k \right) + \sigma^2 \Delta p_t(\theta) \\ &= \nabla \cdot \left(p_t(\theta) \mathbb{E}_{\theta_k \sim p_{\eta k|t}} [U_2(\theta_k)|\theta] \right) + \sigma^2 \Delta p_t(\theta) \\ &= \nabla \cdot (p_t(\theta) \cdot V_t(\theta)) + \sigma^2 \Delta p_t(\theta) \quad (\text{where } V_t(\theta) = \mathbb{E}_{\theta_k \sim p_{\eta k|t}} [U_2(\theta_k)|\theta]) \end{aligned}$$

□

D.2 Proofs for Section 3.2: Privacy erosion in tracing diffusion

Lemma 7 (Leibniz integral rule). *Suppose $f_t : \mathbb{R}^d \rightarrow \mathbb{R}$ is Lebesgue-integrable for each $t \geq 0$. If for almost all $\theta \in \mathbb{R}^d$, the derivative $\frac{df_t}{dt}$ exists and there exists an integrable function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\left| \frac{df_t}{dt}(\theta) \right| \leq g(\theta)$ for all $t \geq 0$ and almost every $\theta \in \mathbb{R}^d$, then*

$$\frac{d}{dt} \int_{\mathbb{R}^d} f_t(\theta) d\theta = \int_{\mathbb{R}^d} \frac{df_t}{dt}(\theta) d\theta, \quad \text{for all } t \geq 0. \quad (49)$$

Lemma 2 (Rate of Rényi divergence). *Let V_t and V'_t be two vector fields on \mathbb{R}^d with $\max_{\theta \in \mathbb{R}^d} \|V_t(\theta) - V'_t(\theta)\|_2 \leq S_v$ for all $t \geq 0$. Then, for corresponding coupled diffusions $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ under V_t and V'_t with noise variance σ^2 , the rate of Rényi divergence at any $t \geq 0$ is upper bounded by*

$$\frac{\partial R_\alpha(\Theta_t \| \Theta'_t)}{\partial t} \leq \frac{1}{\gamma} \frac{\alpha S_v^2}{4\sigma^2} - (1 - \gamma) \sigma^2 \alpha \frac{I_\alpha(\Theta_t \| \Theta'_t)}{E_\alpha(\Theta_t \| \Theta'_t)}. \quad (50)$$

where $\gamma > 0$ is a tuning parameter that we can fix arbitrarily according to our need.

Proof. For brevity, let the functions $R(\alpha, t) = R_\alpha(p_t \| p'_t)$, $E(\alpha, t) = E_\alpha(p_t \| p'_t)$, and $I(\alpha, t) = I_\alpha(p_t \| p'_t)$. Under the stated assumptions $\frac{\partial E(\alpha, t)}{\partial t}$ is bounded as follows.

$$\frac{\partial E(\alpha, t)}{\partial t} = \frac{\partial}{\partial t} \int_{\mathbb{R}^d} \frac{p_t^\alpha}{p_t'^{\alpha-1}} d\theta \quad (51)$$

By Leibniz integral rule (Lemma 7), we exchange order of derivative and integration in (51). The necessary conditions are satisfied because of the following properties about p_t and p'_t :

1. p_t and p'_t have the same support, and their Rényi divergence is well-defined.
2. The distributions of coupled tracing diffusions $\{\theta_t\}_{\eta k < t < \eta(k+1)}$ and $\{\theta'_t\}_{\eta k < t < \eta(k+1)}$ have full support and smooth densities p_t and p'_t (due to convolution with Gaussian noise).
3. The evolutions of probability densities p_t and p'_t with regard to time t satisfy the Fokker-Planck equations (8).

Therefore, we obtain:

$$\begin{aligned}
\frac{\partial E(\alpha, t)}{\partial t} &= \alpha \int_{\mathbb{R}^d} \frac{\partial p_t}{\partial t} \left(\frac{p_t}{p'_t} \right)^{\alpha-1} d\theta - (\alpha-1) \int_{\mathbb{R}^d} \frac{\partial p'_t}{\partial t} \left(\frac{p_t}{p'_t} \right)^\alpha d\theta && \text{(By Lemma 7)} \\
&= \alpha \int_{\mathbb{R}^d} (\sigma^2 \Delta p_t + \nabla \cdot (p_t V_t)) \left(\frac{p_t}{p'_t} \right)^{\alpha-1} d\theta \\
&\quad - (\alpha-1) \int_{\mathbb{R}^d} (\sigma^2 \Delta p'_t + \nabla \cdot (p'_t V'_t)) \left(\frac{p_t}{p'_t} \right)^\alpha d\theta && \text{(From (47))} \\
&= \underbrace{\sigma^2 \alpha \int_{\mathbb{R}^d} \left(\frac{p_t}{p'_t} \right)^{\alpha-1} \Delta p_t d\theta - \sigma^2 (\alpha-1) \int_{\mathbb{R}^d} \left(\frac{p_t}{p'_t} \right)^\alpha \Delta p'_t d\theta}_{\stackrel{\text{def}}{=} F_1} \\
&\quad + \underbrace{\alpha \int_{\mathbb{R}^d} \left(\frac{p_t}{p'_t} \right)^{\alpha-1} \nabla \cdot (p_t V_t) d\theta - (\alpha-1) \int_{\mathbb{R}^d} \left(\frac{p_t}{p'_t} \right)^\alpha \nabla \cdot (p'_t V'_t) d\theta}_{\stackrel{\text{def}}{=} F_2}
\end{aligned}$$

We simplify F_1 as following:

$$\begin{aligned}
F_1 &= \sigma^2 (\alpha-1) \int_{\mathbb{R}^d} \left\langle \nabla \left(\frac{p_t}{p'_t} \right)^\alpha, \nabla p'_t \right\rangle d\theta - \sigma^2 \alpha \int_{\mathbb{R}^d} \left\langle \nabla \left(\frac{p_t}{p'_t} \right)^{\alpha-1}, \nabla p_t \right\rangle d\theta && \text{(From (45))} \\
&= \sigma^2 \alpha (\alpha-1) \int_{\mathbb{R}^d} \left(\frac{p_t}{p'_t} \right)^{\alpha-2} \left\langle \nabla \frac{p_t}{p'_t}, \frac{p_t}{p_t'^2} \nabla p'_t - \frac{\nabla p_t}{p'_t} \right\rangle p'_t d\theta \\
&= -\sigma^2 \alpha (\alpha-1) \mathbb{E}_{\frac{p_t}{p'_t}} \left[\left(\frac{p_t}{p'_t} \right)^{\alpha-2} \left\| \nabla \frac{p_t}{p'_t} \right\|_2^2 \right] && (\because \nabla \frac{p_t}{p'_t} = \frac{\nabla p_t}{p'_t} - \frac{p_t}{p_t'^2} \nabla p'_t) \\
&= -\sigma^2 \alpha (\alpha-1) I(\alpha, t) && \text{(From (28))}
\end{aligned}$$

We upper bound F_2 as following:

$$\begin{aligned}
F_2 &= -\alpha \int_{\mathbb{R}^d} \left\langle \nabla \left(\frac{p_t}{p'_t} \right)^{\alpha-1}, p_t V_t \right\rangle d\theta + (\alpha-1) \int_{\mathbb{R}^d} \left\langle \nabla \left(\frac{p_t}{p'_t} \right)^\alpha, p'_t V'_t \right\rangle d\theta && \text{(From (44))} \\
&= \alpha (\alpha-1) \int_{\mathbb{R}^d} \left(\frac{p_t}{p'_t} \right)^{\alpha-2} \left\langle \nabla \frac{p_t}{p'_t}, \frac{p_t}{p'_t} (V'_t - V_t) \right\rangle p'_t d\theta \\
&\leq \gamma \alpha (\alpha-1) \sigma^2 \int_{\mathbb{R}^d} \left(\frac{p_t}{p'_t} \right)^{\alpha-2} \left\| \nabla \frac{p_t}{p'_t} \right\|_2^2 p'_t d\theta && \text{(From (46) with } b = 2\gamma\sigma^2) \\
&\quad + \frac{\alpha(\alpha-1)S_v^2}{4\gamma\sigma^2} \int_{\mathbb{R}^d} \left(\frac{p_t}{p'_t} \right)^{\alpha-2} \times \left(\frac{p_t}{p'_t} \right)^2 p'_t d\theta && (\because \max_{\theta \in \mathbb{R}^d} \|V_t(\theta) - V'_t(\theta)\|_2 \leq S_v) \\
&= \gamma \sigma^2 \alpha (\alpha-1) I(\alpha, t) + \frac{1}{\gamma} \frac{\alpha(\alpha-1)S_v^2}{4\sigma^2} E(\alpha, t) && \text{(From (26) \& (28))}
\end{aligned}$$

Therefore, we get the following bound on the rate of Rényi divergence:

$$\begin{aligned}
\frac{\partial R(\alpha, t)}{\partial t} &= \frac{1}{\alpha-1} \times \frac{1}{E(\alpha, t)} \times \frac{\partial E(\alpha, t)}{\partial t} \\
&\leq -(1-\gamma) \sigma^2 \alpha \frac{I(\alpha, t)}{E(\alpha, t)} + \frac{1}{\gamma} \frac{\alpha S_v^2}{4\sigma^2}
\end{aligned}$$

□

Discussions about the terms in Lemma 2 Lemma 2 bounds the rate of privacy loss with various terms. Generally speaking, the term $\frac{\alpha S_v^2}{4\sigma^2}$ bounds the worst-case privacy loss growth due to noisy gradient update when $S_v = \frac{S_g}{n}$, while the term $\frac{I_\alpha(\Theta_t \parallel \Theta'_t)}{E_\alpha(\Theta_t \parallel \Theta'_t)}$ amplifies our bound for the rate of privacy loss, as the Rényi privacy loss accumulates during the process. We offer more explanations as the following.

1. $\frac{\alpha S_g^2}{4\sigma^2 n^2}$: This is the first term in the right hand side of (15). It quantifies the worst-case privacy loss due of one noisy gradient update in noisy GD Algorithm 1. The term $\frac{S_g}{n}$ is the sensitivity of average loss gradient $\mathcal{L}_D(\theta)$ over two neighboring datasets D, D' . The larger S_g is, the further apart the parameters θ and θ' after the gradient descent updates on two neighboring dataset D, D' could be, where $\theta = \theta_0 - \eta \nabla \mathcal{L}_D(\theta_0)$ and $\theta' = \theta_0 - \eta \nabla \mathcal{L}_{D'}(\theta_0)$. The term σ^2 is the variance of Gaussian noise. Because additive noise shrink the expected trajectory difference between θ and θ' in noisy GD updates, the larger σ^2 is, the more indistinguishable the distributions of sum of θ, θ' and Gaussian noise will be, therefore the smaller the privacy loss (Rényi divergence between end distributions) will be.
2. $\frac{I_\alpha(\Theta_t \|\Theta'_t)}{E_\alpha(\Theta_t \|\Theta'_t)}$: This term is the second term in the right hand side of (10), which originates from the derivative of p_t, p'_t with regard to time t . To obtain the expression I_α/E_α , we are using the Fokker Planck equation to replace the terms related to $\frac{\partial p_t}{\partial t}, \frac{\partial p'_t}{\partial t}$ with terms determined by the gradient and Laplacian of p_t, p'_t over θ .
 The term $I_\alpha(\Theta_t \|\Theta'_t)$ is the **Rényi information** defined in Definition 2.2., which equals $\mathbb{E}_{\theta \sim p'_t} \left[\left\| \nabla \log \frac{p_t(\theta)}{p'_t(\theta)} \right\|_2^2 \left(\frac{p_t(\theta)}{p'_t(\theta)} \right)^\alpha \right]$. The term $E_\alpha(\Theta_t \|\Theta'_t)$ is the **moment of likelihood ratio** defined in Definition 2.1., which equals $\mathbb{E}_{\theta \sim p'_t} \left[\left(\frac{p_t(\theta)}{p'_t(\theta)} \right)^\alpha \right]$. These two terms differ by a **multiplicative ratio** $\left\| \nabla \log \frac{p_t(\theta)}{p'_t(\theta)} \right\|_2^2$ for their quantities inside expectation. This ratio characterizes the variation of log likelihood ratio function across θ , where θ is taken from distribution p'_t . This is intuitive in the one dimensional case, because $\int_{\theta_1}^{\theta_2} \nabla \log \frac{p_t(\theta)}{p'_t(\theta)} d\theta = \log \frac{p_t(\theta_2)}{p'_t(\theta_2)} - \log \frac{p_t(\theta_1)}{p'_t(\theta_1)}$. Meanwhile since $p_t(\theta), p'_t(\theta)$ are continuous and $\int p_t(\theta) d\theta = \int p'_t(\theta) d\theta = 1$, by mean value theorem, there exists $\tilde{\theta} \in \mathbb{R}^d$ such that the log likelihood ratio $\log \frac{p_t(\tilde{\theta})}{p'_t(\tilde{\theta})}$ is zero. Therefore the variation of log likelihood ratio across θ implicitly increases the largest log likelihood ratio $\max_{\theta \in \mathbb{R}^d} \left[\log \left(\frac{p_t(\theta)}{p'_t(\theta)} \right) - \log \left(\frac{p_t(\tilde{\theta})}{p'_t(\tilde{\theta})} \right) \right] = \max_{\theta \in \mathbb{R}^d} \left[\log \left(\frac{p_t(\theta)}{p'_t(\theta)} \right) \right]$ across θ , which reflects the Rényi privacy loss R_α .
 As a result, intuitively, under some conditions, the larger the Rényi privacy loss R_α is, the larger the variation of log likelihood ratio across θ will be, and therefore the larger the term $\frac{I_\alpha(\Theta_t \|\Theta'_t)}{E_\alpha(\Theta_t \|\Theta'_t)}$ will be. Therefore when the Rényi privacy loss R_α is large, the bound for the rate of privacy loss in (10) Lemma 2 will also be smaller (under $(1 - \gamma) > 0$).
3. γ is a tuning constant to balance the privacy growth rate estimated using the above two terms, thus helping us tune the privacy loss accumulation. See the tightness results in Appendix E for more details.

Theorem 1 (Linear Rényi divergence bound). *Let V_t and V'_t be two vector fields on \mathbb{R}^d , with $\max_{\theta \in \mathbb{R}^d} \|V_t(\theta) - V'_t(\theta)\|_2 \leq S_v$ for all $t \geq 0$. Then, the diffusion under vector fields V_t and V'_t with noise variance σ^2 for time T has α -Rényi divergence of output distributions bounded by $\varepsilon = \frac{\alpha S_v^2 T}{4\sigma^2}$.*

Proof. Setting $\gamma = 1$ in Lemma 2 gives constant privacy loss rate. Integrating over t suffices. \square

Controlling Rényi privacy loss rate under isoperimetry

Lemma 3 ([22] c -LSI in terms of Rényi Divergence). *Suppose $\Theta_t, \Theta'_t \in \mathbb{R}^d$ are random variables such that probability density ratio between Θ_t and Θ'_t lies in $\mathcal{F}_{\Theta'_t}$. Then for any $\alpha \geq 1$,*

$$R_\alpha(\Theta_t \|\Theta'_t) + \alpha(\alpha - 1) \frac{\partial R_\alpha(\Theta_t \|\Theta'_t)}{\partial \alpha} \leq \frac{\alpha^2}{2c} \frac{I_\alpha(\Theta_t \|\Theta'_t)}{E_\alpha(\Theta_t \|\Theta'_t)}, \quad (52)$$

if and only if Θ' satisfies c -LSI.

Proof. Let p and p' denote the probability density functions of Θ_t and Θ'_t respectively. For brevity, let the functions $R(\alpha) = R_\alpha(\Theta_t \|\Theta'_t)$, $E(\alpha) = E_\alpha(\Theta_t \|\Theta'_t)$, and $I(\alpha) = I_\alpha(\Theta_t \|\Theta'_t)$. Let function

$g^2(\theta) = \left(\frac{p(\theta)}{p'(\theta)}\right)^\alpha$. Then,

$$\mathbb{E}_{p'}[g^2] = \mathbb{E}_{p'} \left[\left(\frac{p}{p'}\right)^\alpha \right] = E_\alpha(p||p'), \quad (\text{From (26)})$$

and,

$$\begin{aligned} \mathbb{E}_{p'}[g^2 \log g^2] &= \mathbb{E}_{p'} \left[\left(\frac{p}{p'}\right)^\alpha \log \left(\frac{p}{p'}\right)^\alpha \right] \\ &= \alpha \frac{\partial}{\partial \alpha} \mathbb{E}_{p'} \left[\int_\alpha \left(\frac{p}{p'}\right)^\alpha \log \left(\frac{p}{p'}\right) d\alpha \right] && (\text{Lebniz's rule}) \\ &= \alpha \frac{\partial}{\partial \alpha} \mathbb{E}_{p'} \left[\left(\frac{p}{p'}\right)^\alpha \right] = \alpha \frac{\partial E(\alpha)}{\partial \alpha}. && (\text{From (26)}) \end{aligned}$$

Moreover, from (28),

$$\mathbb{E}_{p'}[\|\nabla g\|_2^2] = \mathbb{E}_{p'} \left[\left\| \nabla \left(\frac{p}{p'}\right)^{\frac{\alpha}{2}} \right\|_2^2 \right] = \frac{\alpha^2}{4} I(\alpha). \quad (53)$$

On substituting the above equalities in (11), we get:

$$\begin{aligned} &\mathbb{E}_{p'}[g^2 \log g^2] - \mathbb{E}_{p'}[g^2] \log \mathbb{E}_{p'}[g^2] \leq \frac{2}{c} \mathbb{E}_{p'}[\|\nabla g\|_2^2] \\ \iff &\alpha \frac{\partial E(\alpha)}{\partial \alpha} - E(\alpha) \log E(\alpha) \leq \frac{\alpha^2}{2c} I(\alpha) \\ \iff &\alpha \frac{\partial \log E(\alpha)}{\partial \alpha} - \log E(\alpha) \leq \frac{\alpha^2}{2c} \frac{I(\alpha)}{E(\alpha)} \\ \iff &\alpha \frac{\partial}{\partial \alpha} ((\alpha - 1)R(\alpha)) - (\alpha - 1)R(\alpha) \leq \frac{\alpha^2}{2c} \frac{I(\alpha)}{E(\alpha)} && (\text{From (24)}) \\ \iff &R(\alpha) + \alpha(\alpha - 1) \frac{\partial R(\alpha)}{\partial \alpha} \leq \frac{\alpha^2}{2c} \frac{I(\alpha)}{E(\alpha)} \end{aligned}$$

□

D.3 Proofs for Section 3.3: Privacy guarantee for Noisy GD

Lemma 4. *Let $\ell(\theta; \mathbf{x})$ be a loss function on closed convex set \mathcal{C} , with a finite total gradient sensitivity S_g . Let $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ be the coupled tracing diffusions for noisy GD on neighboring datasets $D, D' \in \mathcal{X}^n$, under loss $\ell(\theta; \mathbf{x})$ and noise variance σ^2 . Then the difference between underlying vector fields V_t and V'_t for coupled tracing diffusions is bounded by*

$$\max_{\theta \in \mathbb{R}^d} \|V_t(\theta) - V'_t(\theta)\|_2 \leq \frac{S_g}{n}, \quad (54)$$

where $V_t(\theta)$ and $V'_t(\theta)$ are time-dependent vector fields on \mathbb{R}^d , defined in Lemma 1.

Proof. By triangle inequality, for any $\theta \in \mathbb{R}^d$,

$$\begin{aligned} \|V_t(\theta) - V'_t(\theta)\|_2 &\leq \|V_t(\theta)\|_2 + \|V'_t(\theta)\|_2 \\ &\leq \frac{1}{2} \mathbb{E}_{\theta_k \sim p_{\eta_k|t}} [\|\nabla \mathcal{L}_D(\theta_k) - \nabla \mathcal{L}_{D'}(\theta_k)\|_2 |\theta|] && (55) \\ &\quad + \frac{1}{2} \mathbb{E}_{\theta'_k \sim p'_{\eta_k|t}} [\|\nabla \mathcal{L}_{D'}(\theta'_k) - \nabla \mathcal{L}_D(\theta'_k)\|_2 |\theta|]. \quad (\text{From Jensen's inequality}) \end{aligned}$$

By definition of total gradient sensitivity, for any θ_k and θ'_k , we have

$$\|\nabla \mathcal{L}_D(\theta_k) - \nabla \mathcal{L}_{D'}(\theta_k)\|_2 \leq \frac{S_g}{n}, \quad \|\nabla \mathcal{L}_{D'}(\theta'_k) - \nabla \mathcal{L}_D(\theta'_k)\|_2 \leq \frac{S_g}{n}.$$

Therefore, by applying this inequality in equation (55) we obtain (54). □

Theorem 2. Let $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ be the tracing diffusion for $\mathcal{A}_{\text{Noisy-GD}}$ on neighboring datasets D and D' , under noise variance σ^2 and loss function $\ell(\theta; \mathbf{x})$. Let $\ell(\theta; \mathbf{x})$ be a loss function on closed convex set \mathcal{C} , with a finite total gradient sensitivity S_g . If for any neighboring datasets D and D' , the corresponding coupled tracing diffusions Θ_t and Θ'_t satisfy c -LSI throughout $0 \leq t \leq \eta K$, then $\mathcal{A}_{\text{Noisy-GD}}$ satisfies (α, ε) Rényi Differential Privacy for

$$\varepsilon = \frac{\alpha S_g^2}{2c\sigma^4\eta^2} (1 - e^{-\sigma^2 c \eta K}). \quad (56)$$

Proof. The RDP evolution equation (15) holds for projected noisy GD during the tracing diffusion in every time piece $\eta k < t < \eta(k+1)$. Therefore, for $\eta k < t < \eta(k+1)$, the following differential inequality holds:

$$\frac{\partial R(\alpha, t)}{\partial t} \leq \frac{1}{\gamma} \frac{\alpha S_g^2}{4\sigma^2\eta^2} - 2(1-\gamma)\sigma^2 c \left[\frac{R(\alpha, t)}{\alpha} + (\alpha-1) \frac{\partial R(\alpha, t)}{\partial \alpha} \right] \quad (57)$$

Let $a_1 = 2(1-\gamma)\sigma^2 c$, $a_2 = \frac{1}{\gamma} \frac{S_g^2}{4\sigma^2\eta^2}$, and $y = \log(\alpha-1)$.

We define the following function $u(t, y)$ based on Rényi divergence.

$$u(t, y) = \begin{cases} \frac{R(e^y+1, \lim_{t \rightarrow \eta k^+} t)}{e^y+1} - \frac{a_2}{a_1} & \text{if } t = \eta k \\ \frac{R(e^y+1, t)}{e^y+1} - \frac{a_2}{a_1} & \text{if } \eta k < t < \eta(k+1) \end{cases} \quad (58)$$

where we denote the limit privacy at start of a step with $R(\alpha, \lim_{t \rightarrow \eta k^+} t) = R_\alpha(\lim_{t \rightarrow \eta k^+} \Theta_t \| \lim_{t \rightarrow \eta k^+} \Theta'_t)$. Then we can include starting time $t = \eta k$ in the time piece for evolution of $u(t, y)$ and re-write (57) as the following:

$$\frac{\partial u}{\partial t} + a_1 u + a_1 \frac{\partial u}{\partial y} \leq 0, \quad \text{when } \eta k \leq t < \eta(k+1), \quad (59)$$

with initial condition

$$u(\eta k, y) = \frac{R(e^y+1, \lim_{t \rightarrow \eta k^+} t)}{e^y+1} - \frac{a_2}{a_1} \quad (60)$$

We introduce auxiliary variables $\tau = t$, and $z = t - \frac{1}{a_1}y$. By defining $v(\tau, z) = u(t, y)$, we get $\frac{\partial v}{\partial \tau} + a_1 v \leq 0$ from (59), with initial condition $v(\eta k, z) = u(\eta k, -a_1(z - \eta k))$. This PDI implies that for every z , the rate of decay of v is proportional to its present value. The solution for this PDI is $v(\tau, z) \leq v(\eta k, z) e^{-a_1(\tau - \eta k)}$. By bringing back the original variables, we have

$$u(t, y) \leq u(\eta k, y - a_1(t - \eta k)) e^{-a_1(t - \eta k)}, \quad \text{when } \eta k \leq t < \eta(k+1). \quad (61)$$

On undoing the substitution $u(t, y)$ with Rényi divergence, via its definition (58) and initial condition (60), we have that for any $\eta k < t < \eta(k+1)$, the following equation holds.

$$\frac{R(\alpha, t)}{\alpha} - \frac{a_2}{a_1} = \left(\frac{R((\alpha-1)^{-a_1(t-\eta k)} + 1, \lim_{t_0 \rightarrow \eta k^+} t_0)}{(\alpha-1) \cdot e^{-a_1(t-\eta k)} + 1} - \frac{a_2}{a_1} \right) \cdot e^{-a_1(t-\eta k)}. \quad (62)$$

On taking the limit $t \rightarrow \eta(k+1)^-$, we have

$$\frac{R(\alpha, \lim_{t \rightarrow \eta(k+1)^-} t)}{\alpha} - \frac{a_2}{a_1} = \left(\frac{R((\alpha-1)^{-a_1\eta} + 1, \lim_{t_0 \rightarrow \eta k^+} t_0)}{(\alpha-1) \cdot e^{-a_1\eta} + 1} - \frac{a_2}{a_1} \right) \cdot e^{-a_1(t-\eta k)}. \quad (63)$$

Meanwhile, the tracing diffusion expression (5) gives us

$$\lim_{t \rightarrow \eta k^+} \Theta_t = \phi(\Pi_{\mathcal{C}}(\lim_{t \rightarrow \eta k^-} \Theta_t)), \quad \text{and} \quad \lim_{t \rightarrow \eta k^+} \Theta'_t = \phi(\Pi_{\mathcal{C}}(\lim_{t \rightarrow \eta k^-} \Theta'_t)), \quad (64)$$

where $\phi(\theta) = \theta - \eta \cdot \frac{1}{2} (\mathcal{L}_D(\theta) + \mathcal{L}_{D'}(\theta))$ is a mapping on parameter set $\mathcal{C} \subseteq \mathbb{R}^d$. This mapping is the same for neighboring dataset D and D' , because its definition only uses the average gradient

between neighboring datasets D and D' . Therefore by post-processing property of Rényi divergence, we have that for any $\alpha > 1$, the following inequality holds.

$$R(\alpha, \lim_{t \rightarrow \eta k^+} t) \leq R(\alpha, \lim_{t \rightarrow \eta k^-} t). \quad (65)$$

Combining the above two inequalities (63) and (65), we immediately have the following recursive equation:

$$\frac{R(\alpha, \lim_{t \rightarrow \eta(k+1)-t})}{\alpha} - \frac{a_2}{a_1} \leq \left(\frac{R((\alpha-1)^{-a_1 \eta} + 1, \lim_{t \rightarrow \eta k^-} t) - \frac{a_2}{a_1}}{(\alpha-1)^{-a_1 \eta} + 1} \right) e^{-a_1 \eta} \quad (66)$$

Repeating this step for $k = 0, \dots, K-1$, we have

$$\frac{R(\alpha, \lim_{t \rightarrow \eta K^-} t)}{\alpha} - \frac{a_2}{a_1} \leq \left(\frac{R(\alpha_0, \lim_{t \rightarrow 0^-} t) - \frac{a_2}{a_1}}{\alpha_0} - \frac{a_2}{a_1} \right) e^{-a_1 \eta K}, \quad (67)$$

for some $\alpha_0 > 1$. Meanwhile, because coupled tracing diffusion have the same start parameter, we have $R(\alpha_0, \lim_{t \rightarrow 0^-} t) = 0$ for any α_0 . Moreover, since projection is post-processing mapping, we have $R(\alpha, \eta K) \leq R(\alpha, \lim_{t \rightarrow \eta K^-} t)$. Therefore, taking the value $a_1 = 2(1-\gamma)\sigma^2 c$, $a_2 = \frac{1}{\gamma} \frac{S_g^2}{4\sigma^2 n^2}$ in (67), we have

$$R(\alpha, \eta K) \leq \frac{\alpha S_g^2}{8\gamma(1-\gamma)c\sigma^4 n^2} (1 - e^{-2(1-\gamma)\sigma^2 c \eta K}). \quad (68)$$

Setting $\gamma = \frac{1}{2}$ suffices to prove the Rényi privacy loss bound in the theorem. \square

Isoperimetry constants for noisy GD To prove that LSI holds for the tracing diffusion for noisy GD, we first note that the diffusion process (5) can be written as composition of Lipschitz mapping and Gaussian noise for any $\eta k < t < \eta(k+1)$. Meanwhile, the projection at the end of a step is 1-Lipschitz mapping. Then, we rely on the following two lemmas that show Lipschitz transformation and Gaussian perturbation of a probability distribution preserve its LSI property.

Lemma 8 (LSI under Lipschitz transformation [15]). *Suppose a probability distribution p on \mathbb{R}^d satisfies LSI with constant $c > 0$. Let $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a differentiable and L -Lipschitz transformation. The push-forward distribution $T_{\#} p$, representing $T(\Theta)$ when $\Theta \sim p$, satisfies LSI with constant $\frac{c}{L^2}$.*

Lemma 9 (LSI under Gaussian convolution [15]). *Suppose a probability distribution p on \mathbb{R}^d satisfies LSI with constant $c > 0$. For $t > 0$, the probability distribution $p * \mathcal{N}(0, 2t\mathbb{I}_d)$ satisfies LSI with constant $(\frac{1}{c} + 2t)^{-1}$. A special case of this is that $\mathcal{N}(0, 2t\mathbb{I}_d)$ satisfies LSI with constant $\frac{1}{2t}$.*

Lemma 5. *If loss function $\ell(\theta; \mathbf{x})$ is λ -strongly convex and β -smooth over a closed convex set \mathcal{C} , the step-size is $\eta < \frac{1}{\beta}$, and initial distribution is $\Theta_0 \sim \Pi_{\mathcal{C}}(\mathcal{N}(0, \frac{2\sigma^2}{\lambda}\mathbb{I}_d))$, then the coupled tracing diffusion processes $\{\Theta_t\}_{t \geq 0}$ and $\{\Theta'_t\}_{t \geq 0}$ for noisy GD on any neighboring datasets D and D' satisfy c -LSI for any $t \geq 0$ with $c = \frac{\lambda}{2\sigma^2}$.*

Proof. We only prove c -LSI for the tracing diffusion process $\{\Theta_t\}_{t \geq 0}$ on dataset D . The proof for $\{\Theta'_t\}_{t \geq 0}$ is similar.

For any $D \in \mathcal{X}^n$, and any $0 < s < \eta$, recall that the update step in tracing diffusion (5) equals the following random mapping:

$$\Theta_{\eta k+s} = \begin{cases} T_s(\Theta_{\eta k}) + \sqrt{2s\sigma^2}\mathbf{Z}, & \text{if } 0 \leq s < \eta \\ \Pi_{\mathcal{C}}(T_s(\Theta_{\eta k}) + \sqrt{2s\sigma^2}\mathbf{Z}), & \text{if } s = \eta \end{cases} \quad (69)$$

where the mapping $T_s(\theta) = \theta - \eta \cdot \frac{1}{2}(\nabla \mathcal{L}_D(\theta) + \nabla \mathcal{L}_{D'}(\theta)) - s \cdot \frac{1}{2}(\nabla \mathcal{L}_D(\theta) - \nabla \mathcal{L}_{D'}(\theta))$. We first show that $T_s(\theta)$ is $(1-\eta\lambda)$ -Lipschitz. For any $w, v \in \mathcal{C}$, we have

$$\begin{aligned} T_s(w) - T_s(v) &= w - v - \frac{\eta+s}{2}[\nabla \mathcal{L}_D(w) - \nabla \mathcal{L}_D(v)] - \frac{\eta-s}{2}[\nabla \mathcal{L}_{D'}(w) - \nabla \mathcal{L}_{D'}(v)] \\ &= w - v - \left[\frac{\eta+s}{2} \nabla^2 \mathcal{L}_D(z) + \frac{\eta-s}{2} \nabla^2 \mathcal{L}_{D'}(z') \right] (w - v) \\ &\quad \text{(for some } z, z' \in \mathcal{C} \text{ by the mid-value theorems)} \\ &= \left(I - \left[\frac{\eta+s}{2} \nabla^2 \mathcal{L}_D(z) + \frac{\eta-s}{2} \nabla^2 \mathcal{L}_{D'}(z') \right] \right) (w - v) \end{aligned}$$

By λ -strong convexity and β -smoothness of loss function $\ell(\theta; \mathbf{x})$ on \mathcal{C} , we prove that $\nabla^2 \mathcal{L}_D(z)$ and $\nabla^2 \mathcal{L}_{D'}(z')$ both have eigenvalues in the range $[\lambda, \beta]$. Since $s < \eta < \frac{1}{\beta}$, all eigenvalues of $I - [\frac{\eta+s}{2} \nabla^2 \mathcal{L}_D(z) + \frac{\eta-s}{2} \nabla^2 \mathcal{L}_{D'}(z')]$ is in $(0, 1 - \eta\lambda)$. So, T_s is $(1 - \eta\lambda)$ -Lipshitz.

Now, using induction we prove p_t satisfies c -LSI for $c = \frac{\lambda}{2\sigma^2}$ for any $t \geq 0$.

Base step: Being a projection of Gaussian with variance $\frac{\lambda}{2\sigma^2}$ in every dimension, Θ_0 satisfies c -LSI with the given constant by Lemma 8 (because projection is 1-Lipschitz) and Lemma 9.

Induction step: Suppose $\Theta_{\eta k}$ satisfies c -LSI with the above constant for some $k \in \mathbb{N}$. Distribution Θ_t for $\eta k < t < \eta(k+1)$ is same as T_s pushover distribution plus gaussian noise distribution, i.e. $\Theta_t = \Theta_{\eta k \# T_s} * \mathcal{N}(0, 2s\sigma^2 \mathbb{I}_d)$ for $s = t - \eta k$. By using Lemma 8 and 9, we get $(\frac{c}{(1-\eta\lambda)^2 + 2s\sigma^2 c})$ -LSI for Θ_t . Since $s < \eta < \frac{1}{\lambda}$, we have

$$(1 - s\lambda)^2 + 2s\sigma^2 c < 1 - s\lambda + 2s\sigma^2 c = 1.$$

Hence, for $\eta k < t < \eta(k+1)$, Θ_t satisfies c' -LSI with constant $c' > c$, which means it also satisfies c -LSI by definition.

By (69), $\Theta_{\eta(k+1)}$ undergoes an additional projection $\Pi_{\mathcal{C}}(\cdot)$. Since projection is a 1-Lipschitz map, by Lemma 8, it preserves c -LSI. So distribution $\Theta_{\eta(k+1)}$ also satisfies c -LSI. \square

E Proofs and discussions for Section 4: Tightness analysis

Theorem 3. *There exist two neighboring datasets $D, D' \in \mathcal{X}^n$, a start distribution p_0 , and a smooth loss function $\ell(\theta; \mathbf{x})$ whose total gradient $g(\theta; D)$ has finite sensitivity S_g on unconstrained convex set $\mathcal{C} = \mathbb{R}^d$, such that for any step-size $\eta < 1$, noise variance $\sigma^2 > 0$, and $K \in \mathbb{N}$, the Rényi privacy loss of $\mathcal{A}_{\text{Noisy-GD}}$ on D, D' is lower-bounded by*

$$R_\alpha(\Theta_{\eta K} \parallel \Theta'_{\eta K}) \geq \frac{\alpha S_g^2}{4\sigma^2 n^2} (1 - e^{-\eta K}). \quad (70)$$

Proof. We give lower bounds for the Rényi DP guarantee of noisy gradient descent algorithm for minimizing any smooth loss function $\ell(\theta; \mathbf{x})$ with finite total sensitivity S_g . We consider the following L_2 -norm squared loss function with bounded data universe.

$$\ell(\theta; \mathbf{x}) = \frac{1}{2} \|\theta - \mathbf{x}\|_2^2, \text{ where } \theta \in \mathbb{R}^d, \mathbf{x} \in \mathbb{R}^d \text{ and } \|\mathbf{x}\|_2 \leq \frac{S_g}{2}. \quad (71)$$

For any dataset $D = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ of size n , and any $\theta \in \mathbb{R}^d$, the loss is

$$\mathcal{L}_D(\theta) = \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \|\theta - \mathbf{x}_i\|_2^2.$$

It is easy to verify that $\mathcal{L}_D(\theta)$ is 1-smooth. The total gradient of D is

$$g(\theta; D) = \sum_{\mathbf{x} \in D} \nabla \ell(\theta; \mathbf{x}) = n\theta - \sum_{\mathbf{x} \in D} \mathbf{x},$$

with a finite sensitivity S_g .

We construct the two neighboring datasets $D, D' \in \mathcal{X}^n$ such that $D = (\mathbf{x}_1, 0^d, \dots, 0^d)$ and $D' = (\mathbf{x}'_1, 0^d, \dots, 0^d)$, where $\mathbf{x}_1, \mathbf{x}'_1 \in \mathcal{X}$ are two records that are S_g distance apart (i.e. $\|\mathbf{x}_1 - \mathbf{x}'_1\|_2 = S_g$).

Under dataset D , we can express the random variable $\Theta_{\eta K}$ at the K 'th iteration of noisy GD using the following recursion with starting parameter $\Theta_0 = 0^d$.

$$\begin{aligned}\Theta_{\eta K} &= (1 - \eta)\Theta_{\eta(K-1)} + \eta \frac{\mathbf{x}_1}{n} + \sqrt{2\eta\sigma^2} \cdot \mathbf{Z}_{K-1} \\ &= (1 - \eta)^K \Theta_0 + \eta \sum_{i=0}^{K-1} (1 - \eta)^i \frac{\mathbf{x}_1}{n} + \sqrt{2\eta\sigma^2} \sum_{i=0}^{K-1} (1 - \eta)^{K-1-i} \mathbf{Z}_i \\ &= \frac{\eta \mathbf{x}_1}{n} \sum_{i=0}^{K-1} (1 - \eta)^i + \sqrt{2\eta\sigma^2 \sum_{i=0}^{K-1} (1 - \eta)^{2i}} \cdot \mathbf{Z} \quad (\text{where } \mathbf{Z}_i, \mathbf{Z} \sim \mathcal{N}(0, \mathbb{I}_d))\end{aligned}$$

A similar recursion can be used for Θ'_K in Noisy GD under dataset D' . Both Θ_K and Θ'_K are Gaussian random variables with variance $2\eta\sigma^2 \sum_{i=0}^{K-1} (1 - \eta)^{2i}$ in each dimension. Thus, we can calculate their exact divergence.

$$\begin{aligned}R_\alpha(\Theta_{\eta K} \parallel \Theta'_{\eta K}) &= \frac{\alpha \cdot \left\| \eta(\mathbf{x}_1 - \mathbf{x}'_1) \sum_{i=0}^{K-1} (1 - \eta)^i \right\|_2^2}{2 \cdot 2\eta\sigma^2 n^2 \sum_{i=0}^{K-1} (1 - \eta)^{2i}} \\ &= \frac{\alpha \eta^2 S_g^2}{4\eta\sigma^2 n^2} \cdot \frac{(1 - (1 - \eta)^K)^2 / \eta^2}{(1 - (1 - \eta)^{2K}) / (\eta(2 - \eta))} \\ &= \frac{\alpha S_g^2}{4\sigma^2 n^2} \cdot \frac{2 - \eta}{1 + (1 - \eta)^K} (1 - (1 - \eta)^K) \\ &\geq \frac{\alpha S_g^2}{4\sigma^2 n^2} (1 - e^{-\eta K})\end{aligned}$$

This inequality concludes the proof. \square

Corollary 2. Given ℓ_2 -norm squared loss function $\ell(\theta; \mathbf{x}) = \frac{1}{2} \|\theta - \mathbf{x}\|_2^2$ on unconstrained convex set $\mathcal{C} = \mathbb{R}^d$ and bounded data domain with range S_g , and initial parameter $\theta_0 = 0^d$, for any two neighboring datasets $D, D' \in \mathcal{X}^n$, step-size η , noise variance σ^2 , and $K \in \mathbb{N}$, the Rényi privacy loss of $\mathcal{A}_{\text{Noisy-GD}}$ on D, D' is upper-bounded by

$$R_\alpha(\Theta_{\eta K} \parallel \Theta'_{\eta K}) \leq \frac{\alpha S_g^2}{(2 - \eta)\sigma^2 n^2} (1 - e^{-\frac{2-\eta}{2} \eta K}). \quad (72)$$

Proof. To use Theorem 2, we still need to verify c -LSI for the tracing diffusion on ℓ_2 -norm squared loss.

We use the explicit expression for tracing diffusion proved in Theorem 3 to prove c -LSI. We utilize the fact that $\Theta_{\eta K}$, the tracing diffusion for L_2 -norm squared loss at discrete update time ηK , is Gaussian with bounded variance $2\eta\sigma^2 \sum_{i=0}^{K-1} (1 - \eta)^{2i} \leq \frac{2\sigma^2}{2-\eta}$ in each dimension. Therefore, based on Lemma 9, which shows the LSI properties of Gaussian distributions, $\Theta_{K\eta}$ satisfies c -LSI with $c = \frac{2-\eta}{2\sigma^2}$. Similarly, by computing the explicit expression for tracing diffusion at time $\eta k < t < \eta(k+1)$, one can verify Θ_t satisfies c -LSI.

Now, we can directly use Theorem 2 to derive an upper-bound for RDP for Noisy GD under L_2 -squared norm loss. \square

Discussion about tightness results Figure 2 shows the gap between this lower bound and our RDP guarantee derived by Corollary 2, under small step-size $\eta = 0.02$. The upper bound is roughly two times larger than the lower bound, which shows tightness of our privacy guarantee up to a rough constant of two. As comparison, we compute and plot the composition-based bound, which grows as fast as the lower bound in early iterations, but linearly grows above the lower bound, and our RDP guarantee, as K increases to $\Omega(\frac{1}{\eta}) \approx 100 \ll n = 5000$. Moreover, the larger the RDP order α is, the smaller the required number of iterations K is for our RDP guarantee to be superior to the composition-based privacy bound.

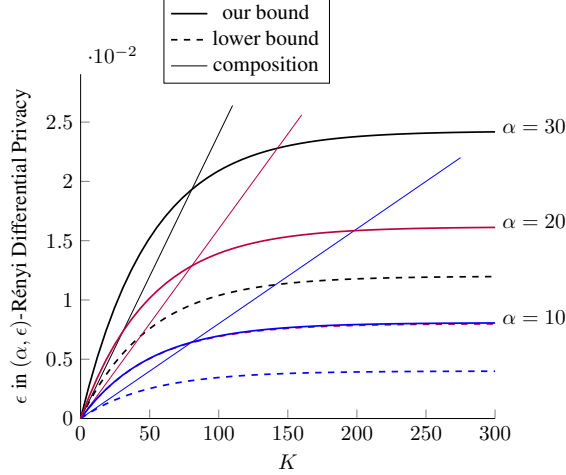


Figure 2: Tightness analysis of our RDP guarantee for the noisy GD algorithm. We show the changes of α -RDP guarantee computed using Corollary 2, over K iterations (number of full passes over the dataset) versus the lower-bounds (dashed lines) which are computed using Theorem 3. The loss function is the ℓ_2 -norm squared function (71), noise standard deviation is $\sigma = 0.02$, the step size is $\eta = 0.02$, the size of the dataset is $n = 5000$, and the finite ℓ_2 -sensitivity for total gradient is $S_g = 4$. The expression for computing the privacy loss in Baseline composition-based analysis (derived by moment accountant [1] with details in Appendix E) is: $\varepsilon = \frac{\alpha S_g^2}{4n^2\sigma^2} \cdot \eta K$

Gap between our upper bound and lower bound There is a gap between the exponent and constant of our privacy upper bound Corollary 2 and the lower bound Theorem 3. We analyze the gap as follows.

1. **The gap in exponent:** There is a $\frac{2-\eta}{2}$ multiplicative gap between the exponent of our privacy upper bound and the lower bound. In hindsight, this is because discretized noisy GD converges to a biased stationary distribution. Therefore, our LSI constant bound $c = \frac{2-\eta}{2\sigma^2}$ depends on the discretization bias caused by step-size η , thus causing the exponent gap in our privacy bound.
2. **The gap in constant:** Our upper bound is larger than the lower bound by roughly a multiplicative constant of two. This is due to the **balancing ratio** $\gamma > 0$ in Lemma 2 for bounding the rate of privacy loss.
 - (a) **At the start of Noisy GD:** setting $\gamma = 1$ in (10) results in a smaller privacy loss rate bound. This is because, at the start of noisy GD, the accumulated privacy loss $R_\alpha(\Theta_t \parallel \Theta'_t)$ is small, thus leading to a small second term I_α/E_α in (10), by Lemma 3. Setting $\gamma = 1$ reduces the coefficient $\frac{1}{\gamma}$ for the dominating first term of (10), at a small cost of increasing the coefficient for the smaller second term I_α/E_α . This facilitates a smaller privacy loss rate bound, and is reflected in the similar growth of composition bound (equivalent to setting $\gamma = 1$) and our lower bound in Figure 2.
 - (b) **As Noisy GD converges:** setting $\gamma \rightarrow 0$ in (10) results in a smaller privacy loss rate bound. This is because, at convergence, the accumulated privacy loss $R_\alpha(\Theta_t \parallel \Theta'_t)$ is larger, thus leading to more significant second term I/E in (10). Setting $\gamma \rightarrow 0$ in (10) reduces the coefficient $-(1-\gamma)$ for the dominating second term I/E , thus facilitate a smaller bound for the privacy loss rate.
3. In our proof for Theorem 2, we set $\gamma = \frac{1}{2}$ to **balance** privacy loss rate estimates at the start and convergence of noisy GD, thus obtaining the smallest privacy bound at convergence, as shown in the proof.

Derivation for Baseline composition-based privacy bound Abadi et al. [1] introduce the moments accountant $\alpha(\lambda)$ for noisy SGD in Eq (2) of their paper, which effectively tracks the scaled Renyi divergence between processes. Therefore in Figure 1, we plot moment accountant bound in Abadi et al. [1] as baseline composition privacy analysis.

1. We first use **moments bound on the Gaussian mechanism** (following Lemma 3 in Abadi et al. [1]) to bound the log moment $\alpha_{\mathcal{M}}(\lambda)$ of data-sensitive computation one update $M : \mathcal{M}(D) = \frac{\eta}{n} \sum_{x_i \in D} \nabla \ell(\theta; x_i) + \mathcal{N}(0, 2\eta\sigma^2 \mathbb{I}_d)$ in our Algorithm 1.

By Eq (2) in Abadi et al. [1], and that $M(D), M(D')$ are Gaussian distributions (with variance $2\eta\sigma^2$ in every dimension and means at most $\frac{\eta}{n} S_g$ apart in ℓ_2 norm), we bound $\alpha_{\mathcal{M}}(\lambda) \leq \frac{\lambda(\lambda+1)\eta S_g^2}{4n^2\sigma^2}$.

2. We then **compose log moment bound for K iterations** by Theorem 2 [Composibility] of log moment bound in Abadi et al. [1], and we obtain $\alpha(\lambda) \leq K \cdot \alpha_{\mathcal{M}}(\lambda) = \frac{K\lambda(\lambda+1)\eta S_g^2}{4n^2\sigma^2}$.
3. Finally by definition of log moment (Eq (2) of Abadi et al. [1]) and Renyi divergence ((1) in our paper), we take $\lambda \leftarrow \alpha - 1$ and $R_\alpha(\Theta_K \parallel \Theta_K) \leftarrow \frac{\alpha(\lambda)}{\lambda}$, and obtain the **baseline composition privacy bound** $\epsilon = \frac{\alpha S^2}{4n^2\sigma^2} \cdot \eta K$ from the log moment bound. We use this expression in Figure 1 and 2.

F Proofs for Section 5: Utility analysis

Theorem 4. For Lipschitz smooth strongly convex loss function $\ell(\theta; \mathbf{x})$ on a bounded closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$, and dataset $D \in \mathcal{X}^n$ of size n , if the step-size $\eta = \frac{\lambda}{2\beta^2}$ and the initial parameter $\theta_0 \sim \Pi_{\mathcal{C}}(\mathcal{N}(0, \frac{2\sigma^2}{\lambda} \mathbb{I}_d))$, then the noisy GD Algorithm 1 is (α, ϵ') -Rényi differentially private, where $\alpha > 1$ and $\epsilon' > 0$, and satisfies

$$\mathbb{E}[\mathcal{L}_D(\theta_{K^*}) - \mathcal{L}_D(\theta^*)] = O\left(\frac{\alpha\beta d L^2}{\epsilon' \lambda^2 n^2}\right), \quad (73)$$

by setting noise variance $\sigma^2 = \frac{4\alpha L^2}{\lambda \epsilon' n^2}$, and number of updates $K^* = \frac{2\beta^2}{\lambda^2} \log\left(\frac{n^2 \epsilon'}{\alpha d}\right)$.

Equivalently, for $\epsilon \leq 2 \log(1/\delta)$ and $\delta > 0$, Algorithm 1 is (ϵ, δ) -differentially private, and satisfies

$$\mathbb{E}[\mathcal{L}_D(\theta_{K^*}) - \mathcal{L}_D(\theta^*)] = O\left(\frac{\beta d L^2 \log(1/\delta)}{\epsilon^2 \lambda^2 n^2}\right), \quad (74)$$

by setting noise variance $\sigma^2 = \frac{8L^2(\epsilon + 2 \log(1/\delta))}{\lambda \epsilon^2 n^2}$, and number of updates $K^* = \frac{2\beta^2}{\lambda^2} \log\left(\frac{n^2 \epsilon^2}{4 \log(1/\delta) d}\right)$.

Proof. From Lemma 6, we have

$$\mathbb{E}[\mathcal{L}_D(\theta_K) - \mathcal{L}_D(\theta^*)] \leq \frac{2\beta L^2}{\lambda^2} e^{-\lambda \eta K} + \frac{2\beta d \sigma^2}{\lambda}. \quad (75)$$

Since $\eta = \frac{\lambda}{2\beta^2} \leq \frac{1}{\beta}$, by Corollary 1, the noisy GD with K iterations will be (α, ϵ') -RDP as long as $\sigma^2 \geq \frac{4\alpha L^2}{\lambda \epsilon' n^2} (1 - e^{-\lambda \eta K/2})$. Therefore, if we set $\sigma^2 = \frac{4\alpha L^2}{\lambda \epsilon' n^2}$, noisy GD is (α, ϵ') -RDP for any K . On substituting this noise variance in (75), we get

$$\mathbb{E}[\mathcal{L}_D(\theta_K) - \mathcal{L}_D(\theta^*)] \leq \frac{2\beta L^2}{\lambda^2} e^{-\lambda \eta K} + \frac{8\alpha L^2 \beta d}{\lambda^2 \epsilon' n^2}. \quad (76)$$

By setting $K^* = \frac{1}{\lambda \eta} \log\left(\frac{\epsilon' n^2}{\alpha d}\right) = \frac{2\beta^2}{\lambda^2} \log\left(\frac{\epsilon' n^2}{\alpha d}\right)$, we can control the empirical risk to be

$$\mathbb{E}[\mathcal{L}_D(\theta_{K^*}) - \mathcal{L}_D(\theta^*)] \leq \frac{10\alpha L^2 \beta d}{\lambda^2 \epsilon' n^2}. \quad (77)$$

Now, we convert the optimal excess risk guarantee under an (α, ϵ') RDP constraint to an optimal excess risk guarantee under (ϵ, δ) DP constraint. Let $\epsilon > 0$ and $0 < \delta < 1$ be two constants such that $\epsilon \leq 2 \log(1/\delta)$. As per DP transition Theorem 7, (α, ϵ') -RDP implies (ϵ, δ) -DP for $\alpha = 1 + \frac{2}{\epsilon} \log(1/\delta)$ and $\epsilon' = \frac{\epsilon}{2}$. By using this conversion, we bound (77) in terms of DP parameters

as

$$\begin{aligned}
\mathbb{E}[\mathcal{L}_D(\theta_{K^*}) - \mathcal{L}_D(\theta^*)] &\leq \frac{10L^2\beta d}{\lambda^2 n^2} \frac{\alpha}{\varepsilon'} \\
&= \frac{10L^2\beta d}{\lambda^2 n^2} \frac{1 + \frac{2}{\varepsilon} \log(1/\delta)}{\frac{\varepsilon}{2}} \\
\because \varepsilon &\leq 2 \log(1/\delta) \leq \frac{10L^2\beta d}{\lambda^2 n^2} \frac{8 \log(1/\delta)}{\varepsilon^2}.
\end{aligned}$$

The amount of noise needed in terms of DP parameters is

$$\begin{aligned}
\sigma^2 &= \frac{4L^2}{\lambda n^2} \frac{\alpha}{\varepsilon'} \\
&= \frac{4L^2}{\lambda n^2} \cdot \frac{1 + \frac{2}{\varepsilon} \log(1/\delta)}{\frac{\varepsilon}{2}}
\end{aligned}$$

The optimal number of updates K^* in terms of DP parameters is bounded as

$$\begin{aligned}
K^* &= \frac{2\beta^2}{\lambda^2} \log\left(\frac{n^2}{d} \cdot \frac{\varepsilon'}{\alpha}\right) \\
&= \frac{2\beta^2}{\lambda^2} \log\left(\frac{n^2}{d} \cdot \frac{\frac{\varepsilon}{2}}{1 + \frac{2}{\varepsilon} \log(1/\delta)}\right) \\
&\leq \frac{2\beta^2}{\lambda^2} \log\left(\frac{n^2}{d} \cdot \frac{\varepsilon^2}{4 \log(1/\delta)}\right).
\end{aligned}$$

□

Lemma 6. For L -Lipschitz, λ -strongly convex and β -smooth loss function $\ell(\theta; \mathbf{x})$ over a closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$, step-size $\eta \leq \frac{\lambda}{2\beta^2}$, and start parameter $\theta_0 \sim \Pi_{\mathcal{C}}(\mathcal{N}(0, \frac{2\sigma^2}{\lambda} \mathbb{I}_d))$, the excess empirical risk of Algorithm 1 is bounded by

$$\mathbb{E}[\mathcal{L}_D(\theta_K) - \mathcal{L}_D(\theta^*)] \leq \frac{2\beta L^2}{\lambda^2} e^{-\lambda\eta K} + \frac{2\beta d\sigma^2}{\lambda}, \quad (78)$$

where θ^* is the minimizer of $\mathcal{L}_D(\theta)$ in the relative interior of convex set \mathcal{C} , and d is the dimension of parameter.

Proof. By the noisy GD update equation we have

$$\theta_{k+1} = \Pi_{\mathcal{C}}(\theta_k - \eta \nabla \mathcal{L}_D(\theta_k) + \sqrt{2\eta\sigma^2} \mathcal{N}(0, \mathbb{I}_d)). \quad (79)$$

From the definition of projection $\Pi_{\mathcal{C}}(\cdot)$, we have:

$$\begin{aligned}
\Pi_{\mathcal{C}}(\theta^* - \eta \nabla \mathcal{L}_D(\theta^*)) &= \arg \min_{\theta \in \mathcal{C}} \|\theta - \theta^* + \eta \nabla \mathcal{L}_D(\theta^*)\|_2^2 \\
&= \arg \min_{\theta \in \mathcal{C}} \|\theta - \theta^*\|_2^2 + 2\eta \langle \theta - \theta^*, \nabla \mathcal{L}_D(\theta^*) \rangle + \eta^2 \|\nabla \mathcal{L}_D(\theta^*)\|_2^2 \\
&\quad \text{(by optimality of } \theta^* \text{ in } \mathcal{C}) \\
&= \arg \min_{\theta \in \mathcal{C}} \|\theta - \theta^*\|_2^2 + \eta^2 \|\nabla \mathcal{L}_D(\theta^*)\|_2^2 \\
&= \theta^*
\end{aligned}$$

Therefore, by combining the above two, and from contractivity of projection $\Pi_{\mathcal{C}}(\cdot)$ [8, Proposition 17] we have

$$\begin{aligned}
\|\theta_{k+1} - \theta^*\|_2^2 &\leq \|\theta_k - \eta \nabla \mathcal{L}_D(\theta_k) + \sqrt{2\eta\sigma^2} \mathcal{N}(0, \mathbb{I}_d) - (\theta^* - \eta \nabla \mathcal{L}_D(\theta^*))\|_2^2 \\
&= \|\theta_k - \theta^*\|_2^2 + \eta^2 \|\nabla \mathcal{L}_D(\theta_k) - \nabla \mathcal{L}_D(\theta^*)\|_2^2 + 2\eta\sigma^2 \|\mathcal{N}(0, \mathbb{I}_d)\|_2^2 \\
&\quad + 2\langle \theta_k - \theta^*, \sqrt{2\eta\sigma^2} \mathcal{N}(0, \mathbb{I}_d) \rangle - 2\eta \langle \nabla \mathcal{L}_D(\theta_k) - \nabla \mathcal{L}_D(\theta^*), \sqrt{2\eta\sigma^2} \mathcal{N}(0, \mathbb{I}_d) \rangle \\
&\quad - 2\eta \langle \theta_k - \theta^*, \nabla \mathcal{L}_D(\theta_k) - \nabla \mathcal{L}_D(\theta^*) \rangle.
\end{aligned}$$

By β -smoothness of \mathcal{L}_D and $\eta = \frac{\lambda}{2\beta^2}$, we have

$$\eta^2 \|\nabla \mathcal{L}_D(\theta_k) - \nabla \mathcal{L}_D(\theta^*)\|_2^2 \leq \eta \lambda \|\theta_k - \theta^*\|_2^2. \quad (80)$$

By strong convexity of \mathcal{L}_D , we have

$$\begin{aligned} \mathbb{E}[\langle \nabla \mathcal{L}_D(\theta_k), \theta_k - \theta^* \rangle] &\geq \mathbb{E}[\mathcal{L}_D(\theta_k) - \mathcal{L}_D(\theta^*)] + \frac{\lambda}{2} \mathbb{E}[\|\theta_k - \theta^*\|_2^2] \\ &\geq \frac{\lambda}{2} \mathbb{E}[\|\theta_k - \theta^*\|_2^2] + \frac{\lambda}{2} \mathbb{E}[\|\theta_k - \theta^*\|_2^2] \\ &\geq \lambda \mathbb{E}[\|\theta_k - \theta^*\|_2^2]. \end{aligned}$$

By taking expectations on the controlling inequality, and plugging the above results, we get

$$\mathbb{E}[\|\theta_{k+1} - \theta^*\|_2^2] \leq (1 - \lambda\eta) \mathbb{E}[\|\theta_k - \theta^*\|_2^2] + 2\eta\sigma^2 d. \quad (81)$$

By β -smoothness,

$$\mathcal{L}_D(\theta_k) - \mathcal{L}_D(\theta^*) \leq \langle \nabla \mathcal{L}_D(\theta^*), \theta_k - \theta^* \rangle + \frac{\beta}{2} \|\theta_k - \theta^*\|_2^2.$$

By the optimality of θ^* in the relative interior of convex set \mathcal{C} and the fact that $\theta_K \in \mathcal{C}$, we prove

$$\langle \nabla \mathcal{L}_D(\theta^*), \theta_K - \theta^* \rangle = 0.$$

Therefore, $\mathcal{L}_D(\theta_K) - \mathcal{L}_D(\theta^*) \leq \frac{\beta}{2} \|\theta_K - \theta^*\|_2^2$. On taking expectation over θ_K , we have

$$\mathbb{E}[\mathcal{L}_D(\theta_K) - \mathcal{L}_D(\theta^*)] \leq \frac{\beta}{2} \mathbb{E}[\|\theta_K - \theta^*\|_2^2].$$

On unrolling the recursion in (81), we have

$$\begin{aligned} \mathbb{E}[\mathcal{L}_D(\theta_K) - \mathcal{L}_D(\theta^*)] &\leq \frac{\beta}{2} (1 - \eta\lambda)^K \mathbb{E}[\|\theta_0 - \theta^*\|_2^2] + 2\beta d \sigma^2 \sum_{k=0}^{K-1} (1 - \eta\lambda)^k \\ &\leq \frac{\beta}{2} e^{-\lambda\eta K} \mathbb{E}[\|\theta_0 - \theta^*\|_2^2] + \frac{2\beta d \sigma^2}{\lambda}. \end{aligned}$$

Since we always have $\|\mathcal{C}\|_2 \leq 2L/\lambda$, we can bound $\mathbb{E}[\|\theta_0 - \theta^*\|_2^2] \leq \frac{4L^2}{\lambda^2}$ as both $\theta_0, \theta^* \in \mathcal{C}$. Therefore, we have

$$\mathbb{E}[\mathcal{L}_D(\theta_K) - \mathcal{L}_D(\theta^*)] \leq \frac{2\beta L^2}{\lambda^2} e^{-\lambda\eta K} + \frac{2\beta d \sigma^2}{\lambda}.$$

□