Visualization of Two-dimensional Interval Type-2 Fuzzy Membership Functions using General Type-2 Fuzzy Membership Functions

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***Abstract*—In this paper, we propose a novel method for visualizing two-dimensional interval type-2 fuzzy membership functions (2-D IT2 FMFs) using one-dimensional general type-2 fuzzy membership functions (1-D GT2 FMFs), and also describe the procedure for extending our method to fuzzy sets representing higher dimensional data. Then we present a type reduction method for mapping 2-D IT2 fuzzy sets into 2-D type-1 fuzzy sets that uses alpha-plane representation of general fuzzy sets. We discuss the problem of "multiple membership values for the same element," which violates set properties, in an IT2 Fuzzy C-means (FCM) algorithm for clustering and propose a solution that uses transformations in the visualization method. These techniques can be applied to applications involving fuzzy sets that represent multidimensional data for proper visualization and type reduction, such as image segmentation, classification and prediction, to name a few.**

***Keywords*—Interval type-2 fuzzy sets, general type-2 fuzzy sets, two-dimensional type-2 fuzzy sets, footprint of uncertainty, type reduction, fuzzy C-means clustering.**

I. Introduction

Type-1 (T1) fuzzy sets (FSs) are used to model uncertainties in data and mimic the process followed by the human mind for processing in fuzzy logic systems (FLSs) [1]. Type-2 (T2) FSs are used to model the uncertainty of the T1 fuzzy membership function (FMF). Over the decades, there have been a wide range of applications of T1 and T2 FSs. Some well-known applications include control systems for navigation [5], robotics [6], and industrial automation [7].

Depending on the application, the input data to an FLS may not always be of a single dimension. For example, while clustering multidimensional data using interval type-2 (IT2) fuzzy C-means (FCM) clustering [11], the input to the defuzzification stage is a multidimensional fuzzy set that needs to be processed to obtain crisp centroids. However, the Karnik-Mendel (KM) algorithm [15] that is used for type reduction is applicable only for one-dimensional (1-D) fuzzy sets. Hence, this gives rise to the need of techniques for visualizing multidimensional IT2 fuzzy sets as multiple 1-D FSs. While there may be multiple methods of visualization multi-dimensional fuzzy sets [17], defuzzification and type reduction should not affect the accuracy of the algorithm. It is easy to visualize a two-dimensional (2-D) T2 FMF as two overlapping 2-D T1 FMF surfaces, as shown in Fig. 1. Hence, in this paper we deal with 2-D IT2 FMFs for the sake of visual explanation.

In Section II, we give formal definitions of 2-D T2 FSs and their membership functions, namely, interval type-2 (IT2) and general type-2 (GT2) FMFs. In Section III, we describe the method for visualizing IT2 FSs representing *n*-D data using multiple GT2 FSs representing *n* 1-D data. We illustrate our method by working with an IT2 FS representing 2-D data, and visualizing it using two 1-D GT2 fuzzy sets. In Section IV, we describe a method of type reduction for 2-D IT2 FSs using GT2 FSs to obtain 2-D T1 FSs. In Section V, we describe the problem of having multiple membership values for the same element in IT2 FCM clustering [11] and demonstrate how our methods can be used to resolve the issue. We conclude by discussing various applications of these techniques.

II. Two-Dimensional Type-2 Fuzzy Sets

A T2 FS with 2-D *primary variable*, denoted as $\widetilde{\mathbf{A}}$, is a bivariate function on the Cartesian product $\mu_{\widetilde{\mathbf{A}}} : \mathbf{X} \times [0,1] \rightarrow [0,1]$, where $\mathbf{X} \subset \mathbb{R}^2$ and its membership function is denoted by $\mu_{\widetilde{\mathbf{A}}}(\mathbf{x}, u)$ where $\mathbf{x} \in \mathbf{X}$ and $u \in U \subset [0,1]$ [2] [3]. In set builder notation, a T2 FS is expressed as

$$\widetilde{\mathbf{A}} = \left\{ \left((\mathbf{x}, u), \mu_{\widetilde{\mathbf{A}}}(\mathbf{x}, u) \right) | \forall \mathbf{x} \in \mathbf{X}, \forall u \in U \right\}, \tag{1}$$

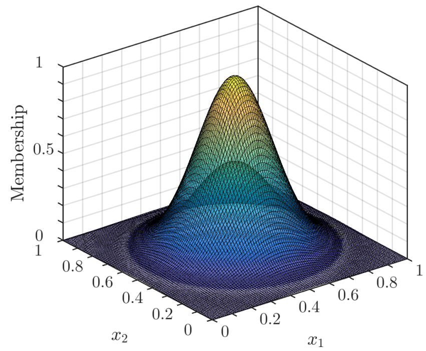

Fig. 1. A typical 2-D T2 FMF.

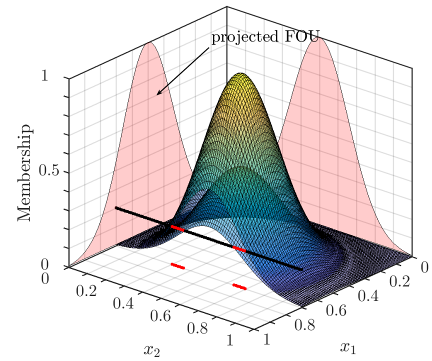

Fig. 2. FOUs projected on the feature axes, along with the interval (shown in red) over which integration is performed to calculate the secondary membership.

where $0 \leq \mu_{\widetilde{\mathbf{A}}}(\mathbf{x}, u) \leq 1$. When $\mu_{\widetilde{\mathbf{A}}}(\mathbf{x}, u) = 1 \quad \forall(\mathbf{x}, u) \in \mathbf{X} \times U$, the fuzzy set is called an IT2 FS, as illustrated in Fig. 1, and is commonly represented as

$$\widetilde{\mathbf{A}} = \int_{\mathbf{x} \in \mathbf{X}} \int_{u \in U} 1 \Big/ (\mathbf{x}, u). \tag{2}$$

An embedded set of a 2-D IT2 FS is a T1 FS expressed as

$$\mathbf{A}_e = \int_{\mathbf{x} \in \mathbf{X}} \mu_{\mathbf{A}_e}(\mathbf{x}) \Big/ \mathbf{x}, \quad \mu_{\mathbf{A}_e}(\mathbf{x}) \in \mu_{\widetilde{\mathbf{A}}}(\mathbf{x}), \tag{3}$$

where $\mu_{\widetilde{\mathbf{A}}}(\mathbf{x})$ is the secondary-set of $\mathbf{x}$ and $\mu_{\mathbf{A}_e}$ is the membership function of the embedded set $\mathbf{A}_e$. In particular, the membership function of the uppermost and the lowermost embedded sets (i.e. the upper membership function (UMF) and the lower membership function (LMF)) of the IT2 FS $\widetilde{\mathbf{A}}$ are expressed as $\overline{\mu}_{\widetilde{\mathbf{A}}}$ and $\underline{\mu}_{\widetilde{\mathbf{A}}}$, respectively.

A T2 FS $\widetilde{\mathbf{G}}$ is called a GT2 FS when it is represented as

$$\widetilde{\mathbf{G}} = \int_{\mathbf{x} \in \mathbf{X}} \int_{u \in U} \mu_{\widetilde{\mathbf{G}}}(\mathbf{x}, u) \Big/ (\mathbf{x}, u), \tag{4}$$

where $\mu_{\widetilde{\mathbf{G}}}(\mathbf{x}, u) \in [0, 1]$, $\mathbf{X} \subset \mathbb{R}^2$ and $U \subset [0, 1]$.

III. Obtaining General Type-2 FMF from 2-D Interval Type-2 FMF

In this section we propose a novel method for visualizing a 2-D IT2 FMF as a pair of GT2 FMFs. That is, we generate two GT2 FMFs, one corresponding to each dimension. The 2-D IT2 FMF $\widetilde{\mathbf{A}}$ is as given in (2) and the 1-D GT2 FMFs $\widetilde{\mathbf{G}}_1$ and $\widetilde{\mathbf{G}}_2$ corresponding to the axes x_1 and x_2, respectively, are given as

$$\widetilde{\mathbf{G}}_i = \int_{x \in \mathbf{X}_i} \int_{u \in U} \mu_{\widetilde{\mathbf{G}}_\mathbf{i}}(x, u) \Big/ (x, u) \quad i = 1, 2, \tag{5}$$

where $\mathbf{X}_i \subset \mathbb{R}$ is an interval on $\mathbb{R}$, and $U \subset [0, 1]$. Since $\mathbf{X}_i$ is an interval, we can define its length as $l(\mathbf{X}_i)$, where $l(\cdot)$ is a length metric.

The upper and lower bounds of the **footprint of uncertainty (FOU)** of the GT2 FMF $\widetilde{\mathbf{G}}_i$ are given by

$$\overline{\mu}_{\widetilde{\mathbf{G}}_i}(x) = \max_{\mathbf{x}_i = x} \ \overline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}) \tag{6}$$

and

$$\underline{\mu}_{\widetilde{\mathbf{G}}_i}(x) = \min_{\mathbf{x}_i = x} \ \underline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}), \tag{7}$$

$\forall \mathbf{x} \in \mathbf{X}$, respectively, where $x \in \mathbf{X}_i$. Each point on the feature domain i ($x \in \mathbf{X}_i$) obtains its upper (lower) membership as the maximum (minimum) of the upper (lower) memberships, of all points on the 2-D domain whose feature i is equal to the point ($x \in \mathbf{X}_i$) being considered. The FOUs thus generated are illustrated in Fig. 2.

The secondary membership of $\widetilde{\mathbf{G}}_1$, $\mu_{\widetilde{\mathbf{G}}_1}(x, u) : \mathbf{X}_1 \times [0, 1] \rightarrow [0, 1]$, is given as the primary membership for each tuple (x, u), $x \in \mathbf{X}_1$, $u \in U$, and can be expressed as

$$\mu_{\widetilde{\mathbf{G}}_1}(x, u) = \frac{1}{l(\mathbf{X}_2)} \int_{\{\mathbf{x}_2 \ | \ \underline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}) \leq u \leq \overline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}), \ \mathbf{x}_1 = x\}} d\mathbf{x}_2. \tag{8}$$

A similar expression can be written for the secondary membership of $\widetilde{\mathbf{G}}_2$. The above equation introduces the 1-D set $\{\mathbf{x}_2 \mid \underline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}) \leq u \leq \overline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}), \ \mathbf{x}_1 = x\} \subset \mathbb{R}$ (illustrated in Fig. 2) over which we integrate a constant to obtain a value representing the length of this set. This set is a collection of points from $\mathbf{X}$, where each point has the same value for feature 1. The set contributes to the primary membership value u of the GT2 FS, since for each point, u lies between its upper membership and lower membership. Thus, the length of this set may be considered as the membership of the primary membership u, or in other words the secondary membership of some $x \in \mathbf{X}_1$. The calculated secondary memberships are illustrated in Fig. 3.

Working with discrete IT2 FMF

For use in practical applications, we give the above explained method for a *discrete* IT2 FMF as well. The feature axes and the primary membership axis are discretized into bins of finite size, and for each bin only the starting point of the bin is considered in forming the GT2 fuzzy sets. We define "bins" over a 1-D domain to be intervals on $\mathbb{R}$ as

$$b_m = [m\epsilon, \ (m+1)\epsilon), \tag{9}$$

where $\epsilon > 0 \in \mathbb{R}$ is the length of each bin. Each bin is indexed with an integer m since over $\mathbb{R}$ the number of bins

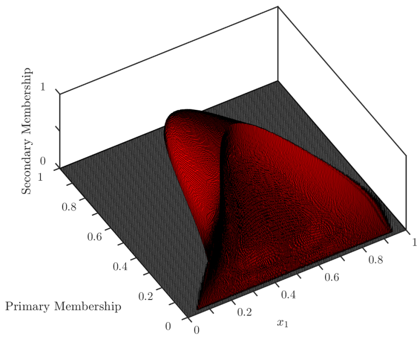

Fig. 3. Secondary memberships over the FOU of the GT2 FS $\widetilde{\mathbf{G}}_i$.

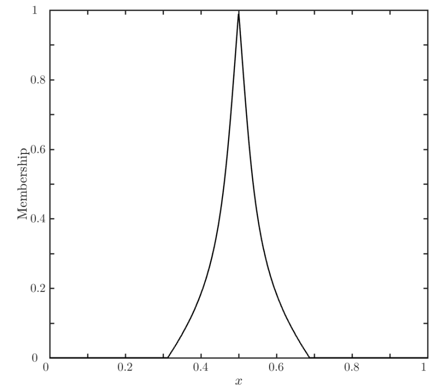

Fig. 4. Centroid of a GT2 FS $\widetilde{\mathbf{G}}_i$ as a type-1 fuzzy set.

are *countably infinite*.

For generating the GT2 FMF i, we project all points of the domain onto the x_i feature axis. To construct the FOU of $\widetilde{\mathbf{G}}_i$, the UMF and LMF for each bin $b_m^i \subset \mathbf{X}_i$ as

$$\mathrm{UMF}(b_m^i) = \max_{\mathbf{x}_i \in b_m^i} \ \overline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}) \tag{10}$$

and

$$\mathrm{LMF}(b_m^i) = \min_{\mathbf{x}_i \in b_m^i} \ \underline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}), \tag{11}$$

$\forall \mathbf{x} \in \mathbf{X}$, where $\mathbf{x}_i$ is feature (dimension) i of $\mathbf{x}$. Note that in the discrete case, the set $\mathbf{X}$ is a discrete set. Each bin on feature i axis obtains its upper (lower) membership as the maximum (minimum) of the upper (lower) memberships, of all points on the 2-D domain whose dimension i lies in that bin. Hence, the upper and lower bounds of the FOU of the GT2 FMF $\widetilde{\mathbf{G}}_i$ are given as

$$\overline{\mu}_{\widetilde{\mathbf{G}}_i}(x) = \mathrm{UMF}(b_m^i) \tag{12}$$

and

$$\underline{\mu}_{\widetilde{\mathbf{G}}_i}(x) = \mathrm{LMF}(b_m^i), \tag{13}$$

respectively, where $x \in \mathbf{X}_i$ and $m = \lfloor x/\epsilon \rfloor$. The set $\mathbf{X}_i$ is a set consisting of $\mathbf{x}_i$, for all $\mathbf{x} \in \mathbf{X}$.

The secondary membership of $\widetilde{\mathbf{G}}_i$, given as $\mu_{\widetilde{\mathbf{G}}_i}(x, u)$: $\mathbf{X}_i \times [0, 1] \rightarrow [0, 1]$, is calculated for discrete values of $u \in [0, 1]$ and $x \in \mathbf{X}_i$ lying between $\overline{\mu}_{\widetilde{\mathbf{G}}_i}(x)$ and $\underline{\mu}_{\widetilde{\mathbf{G}}_i}(x)$ using the following method, which resembles the continuous case.

We divide the primary membership axis (U) into bins and give the secondary membership of $\widetilde{\mathbf{G}}_i$, $\mu_{\widetilde{\mathbf{G}}_i}(x, u)$, where $u \in U$, as the T1 memberships of these bins. A bin of size δ over the primary membership axis is given by the expression h_j, where $j \in [0, 1/\delta)$ and j is an iterator over the bins. Bin h_j is a subset of $[0, 1]$, and is given as

$$h_j = [j\delta, \ (j + 1)\delta), \tag{14}$$

where $\delta > 0 \in \mathbb{R}$ is the length of each bin.

We obtain the secondary membership of $\widetilde{\mathbf{G}}_1$, or the primary membership for each tuple (x, u), $x \in \mathbf{X}_1$ and $u \in U$, to be same for all x and u belonging to the same bin combination b_m and h_j. The secondary membership is given as

$$\mu_{\widetilde{\mathbf{G}}_1}(x, u) = \frac{\sum_{\{\mathbf{x} \ | \ \mathbf{x} \in b_m^1, u \in h_j, \underline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}) \leq u \leq \overline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x})\}} 1}{\sum_{\{\mathbf{x} \ | \ \mathbf{x} \in b_m^1\}} 1}, \tag{15}$$

where $m = \lfloor x/\epsilon \rfloor$ and $j = \lfloor u/\delta \rfloor$. A similar expression can be written for the secondary membership of $\widetilde{\mathbf{G}}_2$. The above expression essentially provides the normalized number of elements in the set $\{\mathbf{x} \mid \mathbf{x} \in b_m^1, u \in h_j, \underline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x}) \leq u \leq \overline{\mu}_{\widetilde{\mathbf{A}}}(\mathbf{x})\}$, and hence the secondary membership.

The above described procedure can be extended to *n*-D IT2 FMFs by generating n 1-D GT2 FMFs, one for each feature axis. In that case, for computing the 1-D GT2 FMF $\widetilde{\mathbf{G}}_i$, the FOU for some $x \in \mathbf{X}_i$ is calculated by using the maximum UMF and minimum LMF of all points of the domain having dimension i as x. The secondary membership is calculated by obtaining the normalized size of a set, similar to that given in (8), containing *n*-D points whose dimension i is fixed.

IV. Type Reduction of IT2 FMF using GT2 FMF

Type reduction refers to the process of finding a proper mapping from a T2 FS to a T1 FS. In this section, we propose a method for type reduction of a 2-D IT2 FS $\widetilde{\mathbf{A}}$ (as given in Fig. 1). We apply the method given in Section III to generate two 1-D GT2 FMFs, $\widetilde{\mathbf{G}}_1$ and $\widetilde{\mathbf{G}}_2$, one corresponding to each feature axis, x_1 and x_2. We then calculate the centroid of $\widetilde{\mathbf{G}}_1$ and $\widetilde{\mathbf{G}}_2$, which are T1 FSs [3] and represented as $C_{\widetilde{\mathbf{G}}_1}$ and $C_{\widetilde{\mathbf{G}}_2}$ respectively, by taking the union of centroids of all the α-planes [4] of IT2 FS [22]. Fig. 4 illustrates the centroid of a GT2 FS.

The type reduced membership function representing the 2-D T1 FS is given as

$$\mu_{\widetilde{\mathbf{A}}}(x_1, x_2) = \min\Big(C_{\widetilde{\mathbf{G}}_1}(x_1), C_{\widetilde{\mathbf{G}}_2}(x_2)\Big). \tag{16}$$

The type-reduced FMF is illustrated in Fig. 5.

In general, when the domain of data-points is *n*-D, the FMF representing the type reduced *n*-D T1 FS is given by

$$\mu_{\widetilde{\mathbf{A}}}(x_1, x_2, ..., x_n) = \min\Big(C_{\widetilde{\mathbf{G}}_1}(x_1), C_{\widetilde{\mathbf{G}}_2}(x_2), ..., C_{\widetilde{\mathbf{G}}_n}(x_n)\Big). \tag{17}$$

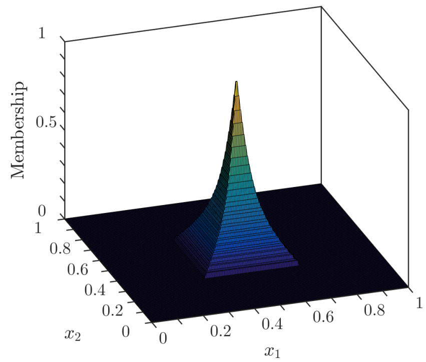

Fig. 5. Type reduced FMF of the 2-D IT2 FMF $\widetilde{\mathbf{A}}$.

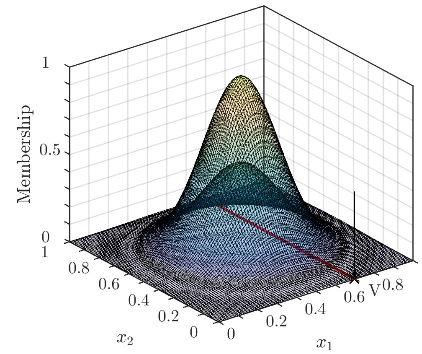

Fig. 6. All points on the red line with their respective membership values being projected to V on the x_1 feature axis.

V. Application

A. Interval Type-2 Fuzzy C-means Clustering

IT2 FCM algorithm [13] [14] is a fuzzy clustering algorithm used to cluster n-D data points into c different clusters. In each iteration of this algorithm, each data-point is assigned c membership values that describe its belongingness to the c clusters, respectively. The algorithm then updates the centroids of the clusters, followed by updating the membership function of each of the c clusters. These updated membership functions are used in the next iteration to compute the c new membership values for the data-points. The update iterations continue till the cluster centres converge (within an ϵ tolerance). The equations for updating the membership values given by 18 and 19.

$$\overline{\mu}_j(\mathbf{x}) = \begin{cases} \frac{1}{\sum_{k=1}^{C}(d_{ji}/d_{ki})^{2/(m1-1)}}, & if \frac{1}{\sum_{k=1}^{C}(d_{ji}/d_{ki})} < \frac{1}{C} \\ \frac{1}{\sum_{k=1}^{C}(d_{ji}/d_{ki})^{2/(m2-1)}}, & otherwise \end{cases} \tag{18}$$

$$\underline{\mu}_j(\mathbf{x}) = \begin{cases} \frac{1}{\sum_{k=1}^{C}(d_{ji}/d_{ki})^{2/(m1-1)}}, & if \frac{1}{\sum_{k=1}^{C}(d_{ji}/d_{ki})} \geq \frac{1}{C} \\ \frac{1}{\sum_{k=1}^{C}(d_{ji}/d_{ki})^{2/(m2-1)}}, & otherwise \end{cases} \tag{19}$$

where i is the index of data-point $\mathbf{x}$, d_{ji} represents the euclidean distance of point i from cluster j, m_1 and m_2 are fuzzifiers that represent different fuzzy degrees. Assigning a single cluster to each data-point is done by hard-partitioning [11], in which the type reduced membership values of the points are compared. If $\mu_j(\mathbf{x}) > \mu_k(\mathbf{x})$ for all $k \in \{1, 2, ..., C\} \setminus j$ then $\mathbf{x}$ is assigned to cluster j.

B. Problem of multiple membership values

The Karnik-Mendel (KM) algorithm requires every element of the IT2 FS to be one-dimensional. Thus, we utilize the KM algorithm independently for each dimension, and the final membership values are determined based on the equations

$$\mu_j(\mathbf{x}) = (\mu_j^R(\mathbf{x}) + \mu_j^L(\mathbf{x}))/2, \quad j = 1, 2, \ldots, C \tag{20}$$

$\forall \mathbf{x} \in \mathbf{X}$, where

$$\mu_j^R(\mathbf{x}) = \sum_{l=1}^{M} \mu_{jl}^R(\mathbf{x})/M \tag{21}$$

and

$$\mu_j^L(\mathbf{x}) = \sum_{l=1}^{M} \mu_{jl}^L(\mathbf{x})/M. \tag{22}$$

Here, M is the total number of features, and $\mu_{jl}^R(\mathbf{x})$ and $\mu_{jl}^L(\mathbf{x})$ are the type reduced values of $\mathbf{x}$ obtained after applying the KM algorithm on the dimension l. As an illustration, for finding $\mu_{j1}(\mathbf{x})$, $\mathbf{x} \in \mathbf{X}$, a new 1-D IT2 FS is created by projecting all data-points onto a 1-D line, which is same as using a transformation which considers only the x_1 feature of each data-point.

In the process of projecting all data-points onto the x_1 axis, all points having the same x_1 coordinate is transformed to the same point on the x_1 axis as shown, even though each of these points can have a different membership value, as illustrated in Fig. 6. In set builder form, the fuzzy set over which KM shall be applied is given as

$$\widetilde{\mathbf{G}}_1 = \sum_{\mathbf{x} \in \mathbf{X}} \sum_{u : \underline{\mu}_{\widetilde{\mathbf{A}}}(x) \leq u \leq \overline{\mu}_{\widetilde{\mathbf{A}}}(x)} 1 \Big/ (\mathbf{x}_1, u) \tag{23}$$

where $\mathbf{x}_i$ represents the feature i of $\mathbf{x} \in \mathbf{X}$, and $\widetilde{\mathbf{A}}$ is the 2-D IT2 FS.

From the definition of fuzzy sets [18], a fuzzy set is associated with a membership function. But the definition of a function: a one to one relation, restricts each element in a fuzzy set to have only one membership value. So, *the set in (23) may not be considered a proper fuzzy set,* and hence KM algorithm cannot be applied for type reduction. This problem is often ignored as the domain of data-points is generally discrete and the probability of two points having the same x_1 coordinate is considered very less. Nevertheless, this problem may have serious implications in the case of continuous domains with dense data-points, such as in image segmentation where the domain data can be very dense.

C. Solution

The above illustrated problem can be solved using the visualization techniques proposed in Section III. As shown in Fig. 6, when points are projected onto the x_1 axis, all points with the same (say V) x_1 coordinate are represented with a single point at V on the x_1 axis, and the membership values of all such points (with x_1 co-ordinate as V) are used in modeling the secondary membership function for $\widetilde{\mathbf{G}}_1$. Hence, $\widetilde{\mathbf{G}}_1$ is no longer an IT2 FS, but a GT2 FS. The fuzzy set formed by considering the projections onto the x_1 axis is given by

$$\widetilde{\mathbf{G}}_1 = \sum_{x \in \mathbf{X}_1} \sum_{u \in U \subset [0,1]} \mu_{\widetilde{\mathbf{G}_1}}(x,u) \Big/ (x,u), \tag{24}$$

where the cardinality of U is given by the value of δ as defined in (14), and $\mu_{\widetilde{\mathbf{G}_1}}(x,u)$ is given by (15).

We can see that the FMF modeled in (24) does not violate the definition of "functions" as the domain of the first summation is $\mathbf{X}_1$ in (24) unlike $\mathbf{X}$ as in (23), and hence is a valid fuzzy set.

VI. Conclusion

In this paper, we presented a way of visualizing 2-D IT2 FS using 1-D GT2 FS. We projected data-points onto the feature axes and all the points that were projected to the same feature value contributed to the secondary membership for that feature. The KM algorithm for GT2 FSs [22] using alpha slice notation was then applied to obtain type reduced 1-D T1 FS. Using the type reduced version of the GT2 FS, we presented a method for type reduction of a 2-D IT2 FS to a 2-D T1 FS.

T2 FS modeling is used in various applications involving classification [9], image segmentation [19], predictive analysis [8], and signal processing [10], to name few. They use type reduction as a part of output processing, but the current procedures face problems when the data involves dense data-points or a continuous domain. We pointed out such a problem in the existing IT2 FCM clustering algorithm [11] [12] [16] and suggested ways of resolving this issue using the new method introduced in this paper. As a part of future research, we intend to work on visualizing 2-D GT2 FSs using multiple 1-D GT2 fuzzy sets.

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