Pumping Lemma: Let $L$ be a regular language. Then there exists a constant $n$ (which depends on $L$) such that for every string $w$ in $L$ satisfying $|w| \geq n$, we can break $w$ into three strings $w = xyz$, such that

(a) $y \neq \epsilon$,
(b) $|xy| \leq n$
(c) For all $k \geq 0$, the string $xy^kz$ is also in $L$. 
Examples:

Let $L = \{a^mb^m \mid m \geq 1\}$.
Then $L$ is not regular.
Proof: Suppose by way of contradiction that $L$ is regular.
Let $n$ be as in the Pumping Lemma.
Let $w = a^n b^n$.
Let $w = xyz$ be as in the Pumping Lemma.
Note that $y$ consists only of $a$s.
Thus, $xy^2z \in L$, however, $xy^2z$ contains more $a$'s than $b$'s.
Examples:

Let \( L = \{ a^i b^j \mid i < j \} \).
Then \( L \) is not regular.
Proof: Suppose by way of contradiction that \( L \) is regular.
Let \( n \) be as in the Pumping Lemma.
Let \( w = a^n b^{n+1} \).
Let \( w = xyz \) be as in the Pumping Lemma.
Note that \( y \) consists only of \( a \)s.
Thus, \( xy^3z \in L \), however, \( xy^3z \) contains more \( a \)'s than \( b \)'s.
Examples:

Let \( L = \{a^p \mid p \text{ is prime}\} \).
Then \( L \) is not regular.

Proof: Suppose by way of contradiction that \( L \) is regular.
Let \( n \) be as in the Pumping Lemma.
Let \( w = a^p \), where \( p \) is prime, and \( p > n \).
Let \( w = xyz \) be as in the Pumping Lemma.
Thus, \( xy^kz \in L \), for all \( k \).
Choose \( k = ? \). Thus, \( xy^kz = a^r \), where \( r \) is not a prime number.
Proof of the Pumping Lemma

Suppose $A = (Q, \Sigma, \delta, q_0, F)$ is a DFA which accepts $L$. Let $n$ be the number of states in $Q$. Suppose $w = a_1a_2\ldots a_n\ldots a_m$ is as given, where $m \geq n$. For $i \geq 1$, let $q_i = \hat{\delta}(q_0, a_1\ldots a_i)$. Then, by Pigeonhole principle, there exists $i, j \leq n, i < j$, such that $q_i = q_j$. Let $x = a_1\ldots a_i, y = a_{i+1}\ldots a_j, z = a_{j+1}\ldots a_m$. As $\hat{\delta}(q_i, y) = q_i$, we have: for all $k, \hat{\delta}(q_i, y^k) = q_i$. Thus, $\hat{\delta}(q_0, xyz) = \hat{\delta}(q_0, xy^kz)$, for all $k$. QED
Closure Properties

- If $L_1, L_2$ are regular, then so is $L_1 \cup L_2$.
- If $L_1, L_2$ are regular, then so is $L_1 \cdot L_2$.
- If $L$ is regular, then so is $\overline{L} = \Sigma^* - L$.
- If $L_1, L_2$ are regular, then so is $L_1 \cap L_2$.
- If $L_1, L_2$ are regular, then so is $L_1 - L_2$.
- If $L$ is regular, then so is $L^R$.
- Let $h$ be a homomorphism. If $L$ is regular, then so is $h(L)$.

Homomorphism: $h(a) \in B^*$, where $B$ is an alphabet set.

$h(\epsilon) = \epsilon$.

$h(a_1a_2\ldots) = h(a_1)h(a_2)\ldots$. 
Decision Problems on Regular Languages

$L = \emptyset$?
$L = \Sigma^*$?
$L(A) = L(A')$?
$w \in L$?