Topics

- Problem modeling by graphs
- Basic graph terminology
- Graph representations
- Graph traversal — BFS and DFS
- Shortest paths — one-to-any (single source) and any-to-any (all pairs)
- (All diagrams will be provided during the lecture.)

Vegetarians and Cannibals Crossing River

- Two vegetarians and two cannibals want to cross a river on a boat that can take only two persons.
- At no time must the vegetarians be outnumbered.
- How should they cross the river?

Vegetarians and Cannibals: Hint

```
                vvccB/

vc/Bvc    vv/Bcc    vvc/Bc    cc/Bvv

     vvcB/c

     c/Bvvc

ccB/vv    vcB/vc

     /Bvvcc
```

Wolf, Goat, Cabbage, Farmer Crossing River

- Wolf eats goat if left alone, goat eats cabbage if left alone.
- The farmer can only take one object with him at a time.
- How should the farmer bring the wolf, goat, and cabbage to the other side of the river?

Wolf, Goat, Cabbage, Farmer Crossing River: Hint

```
     FWGC/

     WC/FG

     FWC/G
C/FWG     W/FGC

FGC/W     FWG/C
     G/FWC

     FG/WC

     /FWGC
```

Decanting: $3 + 5 \to 1$

- A 3-unit bottle and a 5-unit bottle are given.
- Each bottle can be filled from a tap, emptied into a sink, poured into another bottle until itself is empty or the other is filled.
- How to obtain 1-unit of liquid?

Decanting: Hint

```
     00
  30     05
35  03     32
    33     02
    15     20
    10     25
    01     34
    31     04
```

Decanting : BFS (To be Discussed Later)

```
#define N       24

int visit[4][6];

int qx[N], qy[N];

int head, tail = 0;
```

```
check( int ox, int oy, int x, int y ) {
  if( visit[x][y] ) return;
  enqueue( x, y );
  visit[x][y] = 1;
  printf( "%d %d -> %d %d\n", ox, oy, x, y );
}
```

```
enqueue( int x, int y ) {
  if( (tail+1)%N == head ) {
    printf( "queue overflow\n" );  exit( 1 );
  }
  qx[tail] = x;  qy[tail] = y;
  if( ++tail >= N ) tail = 0;
}

dequeue( int *x, int *y ) {
  if( head == tail ) {
    printf( "queue underflow\n" );  exit( 1 );
  }
  *x = qx[head];  *y = qy[head];
  if( ++head >= N ) head = 0;
}
```

```
main() {
  int i, j, x, y;

  for( i = 0;  i < 4;  i++ )
    for( j = 0;  j < 6;  j++ )
      visit[i][j] = 0;

  check( 0, 0, 0, 0 );
```

```
  while( head != tail ) {
    dequeue( &x, &y );
    if( x < 3 ) check( x,y, 3, y ); // fill x
    if( x < 3 && y >= 3-x ) check( x,y, 3, y-3+x );
    if( y > 0 && y < 3-x ) check( x,y, x+y, 0 );
    if( x > 0 ) check( x,y, 0, y ); // empty x

    if( y < 5 ) check( x,y, x, 5 ); // fill y
    if( y < 5 && x >= 5-y ) check( x,y, x-5+y, 5 );
    if( x > 0 && x < 5-y ) check( x,y, 0, y+x );
    if( y > 0 ) check( x,y, x, 0 ); // empty y
  }
}
```

```
0 0 -> 0 0
0 0 -> 3 0
0 0 -> 0 5
3 0 -> 3 5
3 0 -> 0 3
0 5 -> 3 2
0 3 -> 3 3
3 2 -> 0 2
3 3 -> 1 5
0 2 -> 2 0
1 5 -> 1 0
2 0 -> 2 5
1 0 -> 0 1
2 5 -> 3 4
0 1 -> 3 1
3 4 -> 0 4
```

Basic Graph Terminology: Vertices and Edges

- Mathematically a graph G consists of a vertex set V and an edge set E:

$$G = (V, E).$$

- An edge has two end-points, each of which must be a vertex.
- A vertex may or may not be an end-point of an edge.
- Unless explicitly specified otherwise, a graph usually means undirected graph.

Incidence

- Incidence is a relation between a vertex and an edge.

- For any vertex v and any edge e,

  vertex v is incident on edge e,

  or

  edge e is incident on vertex v,

  if (and only if)

  v is an end-point of e.

Vertex Adjacency

- For any distinct vertices u and v,

  u and v are adjacent

  if (and only if)

  they are incident on the same edge.

- Two distinct vertices are adjacent if (and only if) they are the end-points of the same edge.

Edge Adjacency

- For any distinct edges e and f,

  e and f are adjacent

  if (and only if)

  they are incident on the same vertex.

- Two distinct edges are adjacent if (and only if) they share an end-point.

Simple Graphs

- An edge is a loop if (and only if) its two end-points are identical.
- Two edges are parallel if (and only if) they have the same end-points.
- A graph is simple if (and only if) there are no loops and parallel edges.

Vertex Degree

- For any vertex v, the degree of v is the number of times edges incident on v.

- If there are μ loops and ν (non-loop) edges incident on a vertex v, the degree of v is
$$d(v) = 2\mu + \nu.$$

Isolated Vertices

- A vertex is isolated if (and only if) its degree is zero.
- That is, a vertex is isolated if (and only if) no edges incident on it; equivalently, it is incident on no edges; or no edges are incident on it.

The Handshake Theorem

- For any graph $G = (V, E)$,

$$\sum_{v \in V} d(v) = 2|E|$$

- Proof:

  Each edge has two end-points.

  $d(v) =$ the number of times v labels an end-point.

  $\sum_{v \in V} d(v) =$ number of end-points.

A Corollary

- The number of odd-degree vertices is even.

- Proof:

$$\sum_{v \in V} d(v) = \sum_{2 | d(v)} d(v) + \sum_{2 \nmid d(v)} d(v) = 2|E|.$$

Applications

- Is it possible that each of a group of nine people knows exactly five others in the group?
- Is it possible to have a graph of five verices of degrees 1, 2, 3, 4, 5?

Paths

- Let v_0 and v_1, not necessarily distinct, be vertices of a graph.

- A path from v_0 to v_1 of length n is an alternating sequence of $n+1$ vertices and n edges of the form:

$$v_0\ e_1\ v_1\ \cdots v_{n-1}\ e_n\ v_n$$

such that e_i is incident on v_{i-1} and v_i for $i = 1, \ldots, n$.

- For a simple graph, since e_i is completely determined by v_i and v_{i+1}, the path may be written simply as

$$v_0\ v_1\ \cdots v_{n-1}\ v_n$$

Cycles

- A cycle is a path of nonzero length from a vertex to itself with **distinct** edges.
- A length 1 cycle is a loop.
- A length 2 cycle is two parallel edges.
- A simple cycle is a cycle with distinct vertices (except the first and last).

Connectedness

- A graph $G = (V, E)$ is connected if (and only if) for any distinct vertices $u \in V$ and $v \in V$, there is a path from u to v.

Acyclicity

- A graph is acyclic if (and only if) it has no cycles.

Trees

- A tree is a connected acyclic graph.

Graph Representation 1: Adjacency Matrices

- Number the n vertices of a graph either from 0 to $n-1$ or from 1 to n.
- The graph can be represented as a matrix $a[\,][\,]$ such that

$$a[i][j] = \text{number of edges incident on vertices } i, j.$$

- For a simple graph, we have

$$a[i][i] = 0$$

and

$$a[i][j] \leq 1.$$

Graph Representation 2: Adjacency Lists

- Number the n vertices of a graph either from 0 to $n-1$ or from 1 to n.
- Create an array $a[\ ]$ of lists.
- Array entry $a[i]$ lists the vertices adjacent to vertex i.
- Alternatively, create a 2-dimensional jagged array $a[\ ][\ ]$:

$$\text{length of } a[i] = d(i).$$

(Very easily done in Java.)

Adjacency Matrices Versus Adjacency Lists

- Storage: $|V|^2$ against $|V| + 2|E|$
- Access: direct against serial

Example

- Represent the graph

$$G = (\{1, 2, 3, 4, 5, 6\}, \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 5\}\})$$

  as an adjacency matrix and as adjacency lists.

- Represent the graph

$$G = (\{1, 2, 3, 4, 5, 6\}, \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\})$$

  where $f(e_1) = \{6\}$, $f(e_2) = \{1, 5\}$, $f(e_3) = \{1, 5\}$, $f(e_4) = \{2, 4\}$, $f(e_5) = \{3, 4\}$, $f(e_6) = \{3, 4\}$, $f(e_7) = \{1, 1\}$, as an adjacency matrix and as adjacency lists.

Breadth First Search (BFS)

- One way of traversing a graph is to do a breadth first search (BFS).
- BFS can be described as follows:

```
visit a vertex
while there is a least recently visited vertex v do
  visit all unvisited vertices adjacent to v
```

BFS Pseudo Code

```
bfs( start ) {
  for( i = 0;  i < n;  i++ ) visit[i] = false;
  visit[start] = true;  show( start );  enqueue( start );
  while( ! empty() ) {
    for( l = adj[dequeue()];  l;  l = l->next ) {
      v = l->vertex;
      if( !visit[v] ) {
        visit[v] = true;  show( v );  enqueue( v );
      }
    }
  }
}
```

BFS: The Decanting Example

- A graph modeling the previous decanting example is given as adjacency lists.
- What is the rooted ordered tree obtained by a breadth first search starting with 00 (both bottles are empty)?
- Note that the order of the ordered tree is determined by the vertex order in the adjacency lists.

```
00:  30,  05
01:  31,  10,  05,  00
02:  32,  20,  05,  00
03:  33,  30,  05,  00
04:  34,  31,  05,  00
05:  35,  32,  00
10:  30,  00,  15,  01
15:  35,  33,  05,  10
20:  30,  00,  25,  02
25:  35,  34,  05,  20
30:  00,  35,  03
31:  01,  35,  04,  30
32:  02,  35,  05,  30
33:  03,  35,  15,  30
34:  04,  35,  25,  30
35:  05,  30
```

Depth First Search

- Another way of traversing a graph is to do a depth first search (DFS).
- DFS can be described as follows:

```
dfs( v ) {
  mark v as visited
  for each unvisited vertex u adjacent to v do
    dfs( u )
}
```

DFS Pseudo Code

```
dfs_init( start ) {
  for( i = 0;  i < n;  i++ ) visit[i] = false;
  dfs( start );
}

dfs( start ) {
  visit[start] = true;  show( start );
  for( l = adj[start];  l;  l = l->next ) {
    v = l->vertex;
    if( !visit[v] ) dfs( v );
  }
}
```

DFS: The Decanting Example

- Consider the adjacency lists of the graph for the decanting example.
- What is the rooted ordered tree obtained by a depth first search starting with 10?

DFS: The Decanting Example Answer

```
          10
          30
      00       03
      05       33
     /  \      15
  35      32
          02
          20
          25
          34
          04
          31
          01
```

A Remark on the Decanting Problem

- The adjacency lists are not created explicitly and statically. They are created on the flight.
- The possible configurations from the configuration (x, y):

$$
\begin{aligned}
(x, y), x < 3 &\quad\rightarrow\quad (3, y) \\
(x, y), x < 3, y \geq 3 - x &\quad\rightarrow\quad (3, y + x - 3) \\
(x, y), y > 0, y < 3 - x &\quad\rightarrow\quad (x + y, 0) \\
(0, y), x > 0 &\quad\rightarrow\quad (0, y)
\end{aligned}
$$

$$\begin{aligned}
(x, y), y < 5 &\quad \rightarrow \quad (x, 5) \\
(x, y), y < 5, x \geq 5 - y &\quad \rightarrow \quad (x + y - 5, 5) \\
(x, y), x > 0, x < 5 - y &\quad \rightarrow \quad (0, x + y) \\
(x, y), y > 0 &\quad \rightarrow \quad (x, 0)
\end{aligned}$$

Weighted Graphs

- A weighted graph G consists of a vertex set V, an edge set E, and a weight function $w : E \to \mathbf{R}$:

$$G = (V, E, w).$$

- Every graph can be treated as a weighted graph by taking

$$w(e) = 1$$

for any edge $e \in E$.

Some Observations on Weighted Graphs

- Many situations can be modeled as weighted graphs.
- For example, the highways connecting cities may be modeled as a weighted graph with highway distance as the weight function.
- The rooted tree built by a breadth first search starting at vertex v gives the shortest path length of all vertices from v: the shortest distance of a vertex at level L is L.
- In other words, BFS solves the 1-to-any (single source) shortest path problem for the specail case when $w(E) = \{1\}$.

Weighted Graph Representations

- A simple weighted graph can be represented as an adjacency matrix:

$$a[i][j] = w(\{i, j\}).$$

- A weighted graph can also be represented as adjacency lists:

$$a[i] = \text{a list of pairs } (v, w(e))$$

where v is a vertex adjacent to i and e is an edge incident on i and v.

Floyd's Algorithm: All-Pairs Shortest Path

```
floyd() {
  for( k = 0;  k < n;  k++ )
    for( i = 0;  i < n;  i++ )
      for( j = 0;  j < n;  j++ )
        if( a[i][k] + a[k][j] < a[i][j] )
          a[i][j] = a[i][k] + a[k][j];
}
```

Floyd's Algorithm: Comments

- Very easy to code.
- Transform the adjacency matrix to an all-pairs shortest path matrix.
- Complexity $\Theta(n^3)$.
- Theory: dynamic programming subproblem structure:

$$a_{i,j}^{(k)} = \min(a_{i,j}^{(k-1)}, a_{i,k}^{(k-1)} + a_{k,j}^{(k-1)}).$$

- Programming: update in place is correct because

$$a_{i,k}^{(k)} = a_{i,k}^{(k-1)},\ a_{k,j}^{(k)} = a_{k,j}^{(k-1)}.$$

Dijkstra's Algorithm for Single-Source Shortest Paths

- This is a greedy algorithm.
- Theory (skipped).

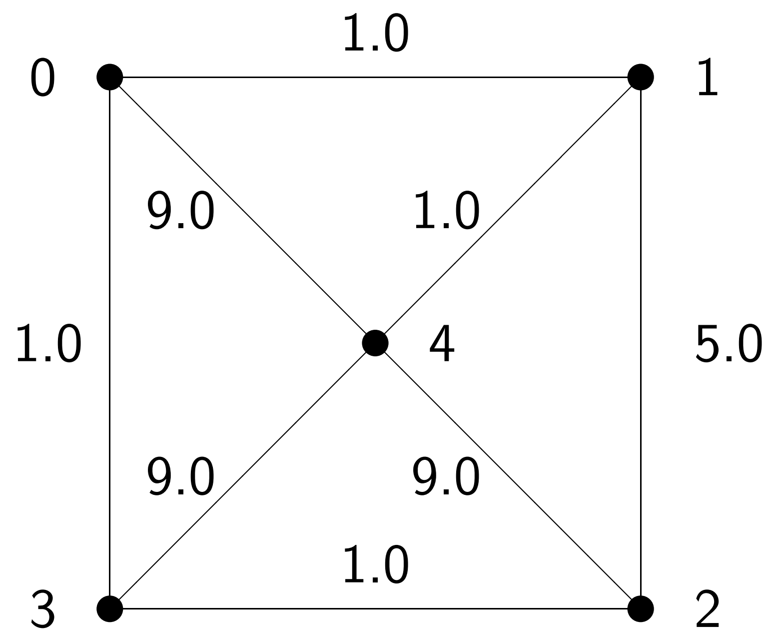
Dijkstra's Algorithm: Pseudo Code

```
for each vertex v do d[v] = MAX_VAL;
d[source] = 0;  p[source] = -1;
do |V| times {
  let d[v] be the smallest among all undeleted vertices
  x = d[v];  delete v;
  for each vertex u adjacent to v do {
    if u is undeleted and x+w[u] < d[u] then {
      d[u] = x + w[u];  p[u] = v;
    }
  }
} // array d gives the shortest distance from source
```

Implementation: Naive Versus Sophisticated

- Two pieces of information: adjacency and distance.
- If the distance information is implemented as an array $d[v]$, coding is simple but may incur a time complexity of $O(|V|^2)$.
- If the distance information is implemented as a priority queue, coding is more involved but the time complexity is $O(|E| \log |V|)$ when the graph is connected.

Dijkstra's Algorithm: Example



Dijkstra's Algorithm: A C Program

```
#include <values.h>

// adjacency nodes

struct edge {
  int  v, wt;
  struct edge *nxt;
};

// adjacency lists

int n, m;
struct edge **adj;
```

```
// priority queue with
// priority updates and
// indirect priorities

int N;
int *delete, *heap, *dist, *where, *parent;
```

```
// build adjacency lists
insert( struct edge ** l, int v, int wt ) {
  struct edge *p, *q;
  p = *l;
  q = (struct edge *) calloc( 1, sizeof(struct edge) );
  q->v = v;
  q->wt = wt;
  q->nxt = p;
  *l = q;
}

show( struct edge *l ) {
  for( ;  l;  l = l->nxt )
    printf( " %d (%d)", l->v, l->wt );
  printf( "\n" );
}
```

```
siftdown( int i ) {
  int  hi, child;

  hi = heap[i];
  while( 2*i+1 <= N-1 ) {
    child = 2*i+1;
    if(child<N-1 && dist[heap[child+1]]<dist[heap[child]])
      child++;
    if( dist[heap[child]] < dist[hi] ) {
      heap[i] = heap[child];  where[heap[i]] = i;
    } else
    break;
    i = child;
  }
  heap[i] = hi;  where[heap[i]] = i;
}
```

```
heapify() {
  int i;

  for( i = N/2-1; i >= 0;  i-- ) siftdown( i );
}

int heap_del() {
  int v;
  v = heap[0];
  delete[v] = 1;
  heap[0] = heap[N-1];
  where[heap[0]] = 0;
  N--;
  siftdown( 0 );
  return v;
}
```

```
heap_decrement( int v, int val ) {
  int i;
  dist[v] = val;
  i = where[v];
  while( i > 0 && val < dist[heap[(i+1)/2 - 1]] ) {
    heap[i] = heap[(i+1)/2 - 1];
    where[heap[i]] = i;
    i = (i+1)/2 - 1;
  }
  heap[i] = v;
  where[heap[i]] = i;
}
```

```
heap_show() {
  int i;

  printf( "*\n" );
  for( i = 0;  i < N;  i++ )
    printf( "%d %d %d\n", i, heap[i], dist[heap[i]] );
}
```

```
dijkstra( int start ) {
  int i, minc, v, u;
  struct edge *e;

  for( i = 0;  i < n;  i++ ) {
    dist[i] = MAXINT;
    heap[i] = i;
    where[i] = i;
    delete[i] = 0;
  }
  N = n;

  dist[start] = 0;  parent[start] = -1;
  heapify();
```

```
  for( i = 0;  i < n;  i++ ) {
    v = heap_del();
    minc = dist[v];
    for( e = adj[v];  e;  e = e->nxt ) {
      u = e->v;
      if( !delete[u] && minc + e->wt < dist[u] ) {
        parent[u] = v;
        heap_decrement( u, minc + e->wt );
      }
    }
    heap_show();
  }
}
```

```
// vertices are numbered from 0

main( int ac, char *av[] ) {
  int i, u, v, wt;

  scanf( "%d %d", &n, &m );  printf( "%d %d\n", n, m );

  adj = (struct edge **) calloc(n, sizeof(struct edge *));

  for( i = 0;  i < m;  i++ ) {
    scanf( "%d %d %d", &u, &v, &wt );
    printf( "%d %d %d\n", u, v, wt );
    insert( &adj[u], v, wt );
    insert( &adj[v], u, wt );
  }
  for( i = 0;  i < n;  i++ ) show( adj[i] );
```

```
  delete = (int *) calloc( n, sizeof(int) );
  parent = (int *) calloc( n, sizeof(int) );
  heap = (int *) calloc( n, sizeof(int) );
  dist = (int *) calloc( n, sizeof(int) );
  where = (int *) calloc( n, sizeof(int) );

  if( ac > 1 )
    dijkstra( atoi(av[1]) );
  else
    dijkstra( 0 );

  for( i = 0;  i < n;  i++ )
    printf( "%d %d %d\n", i, dist[i], parent[i] );
}
```

Dijkstra's Algorithm: Output

```
5 8
0 1 1
0 3 1
0 4 9
1 4 1
1 2 5
4 3 9
4 2 9
3 2 1
```

```
4 (9) 3 (1) 1 (1)
2 (5) 4 (1) 0 (1)
3 (1) 4 (9) 1 (5)
2 (1) 4 (9) 0 (1)
2 (9) 3 (9) 1 (1) 0 (9)
```

```
*
0 1 1
1 3 9
2 2 9
3 0 9
*
0 0 2
1 3 9
2 2 6
*
0 3 3
1 2 6
*
0 2 4
*
```

```
0 2 1
1 1 4
2 4 3
3 3 0
4 0 -1
```

Exercises

1. A complete graph is a simple graph in which any two distinct vertices are adjacent. A complete graph of n vertices is denoted K_n. Describe the rooted ordered tree produced by a bfs and a dfs on K_n.

2. Code a naive Dijkstra's algorithm to run the given example. (By naive we mean using an array instead of a priority queue to store the distance information.)

3. What is the role of $where[\,]$ array in the given Dijkstra's algorithm?

4. Implement Floyd's algorithm to run the given example.