Planar Graphs, Polygons and Triangulations

Lecture 3, CS 4235
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Tutorials

- preparation
  - you are not expected to be able to solve all the exercises
  - the most important thing is that you *try* to solve them
  - exercises with star are difficult
- you can write down your answers and pass them to me
  - I will mark them
  - but these marks will not count towards your final grade

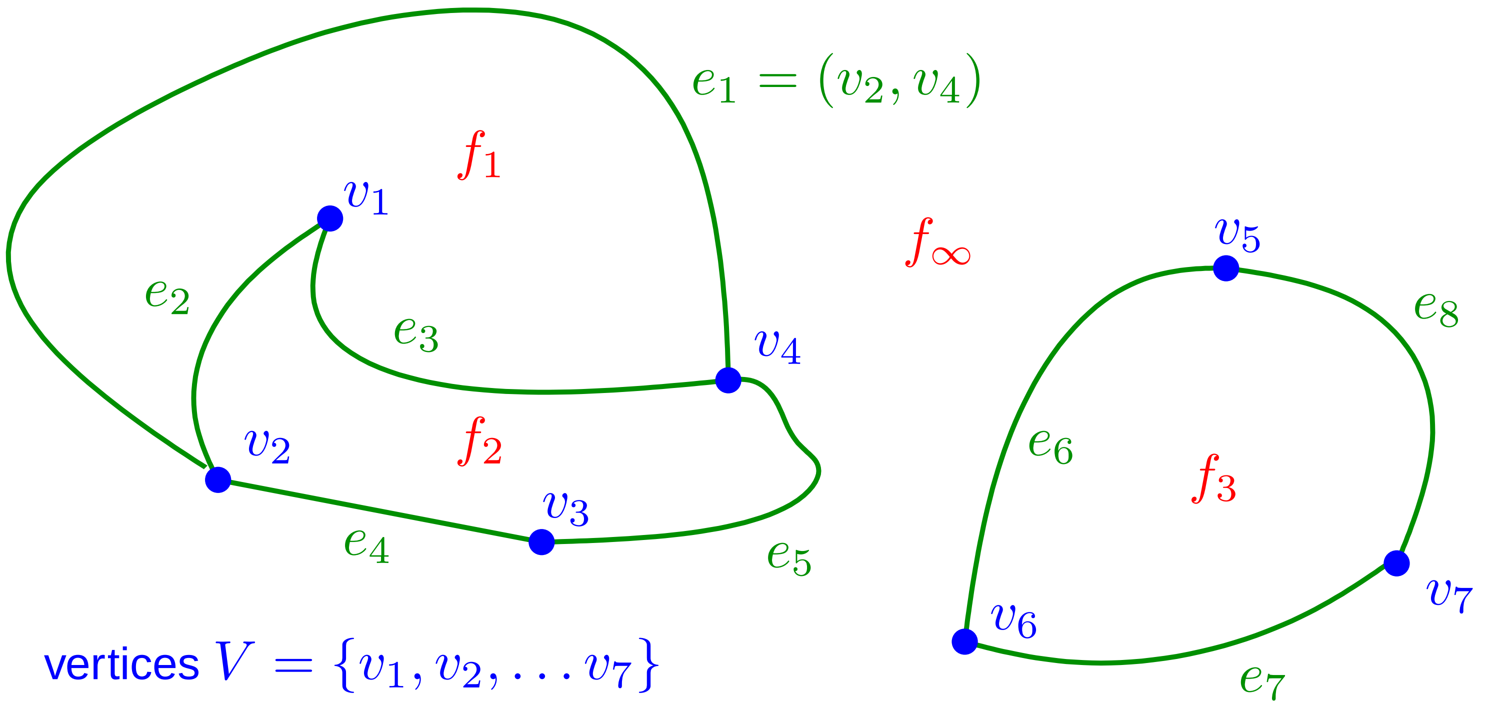
Outline

- reference: Dave Mount lecture notes, lectures 6 and 7
- planar graphs
  - straight line planar graphs
  - trapezoidal map
  - polygons
- triangulation
  - existence
  - algorithm

Planar Graphs

Definition

- graph embedded in $\mathbb{R}^2$
- edges do not intersect in their interior



vertices $V = \{v_1, v_2, \ldots v_7\}$

edges $E = \{e_1, e_2, \ldots e_8\}$

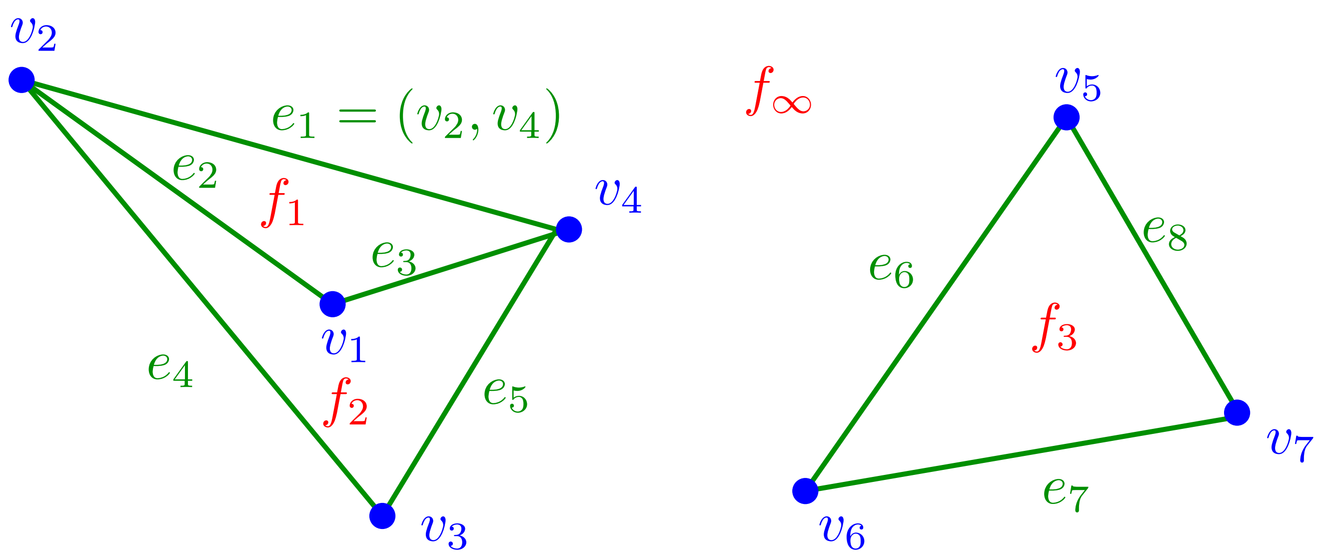
faces $F = \{f_1, f_2, f_3, f_\infty\}$

Properties of planar graphs

- 1 infinite face (f_∞)
- Euler relation:
  - connected planar graph $|V| - |E| + |F| = 2$
  - c connected components $|V| - |E| + |F| - c = 1$
  - proof?
- Theorem: $|E| \leq 3(|V| - 2)$ and $|F| \leq 2(|V| - 2)$
  - proof page 26 of D. Mount lecture notes

Properties (2)

- every planar graph has a straight line embedding

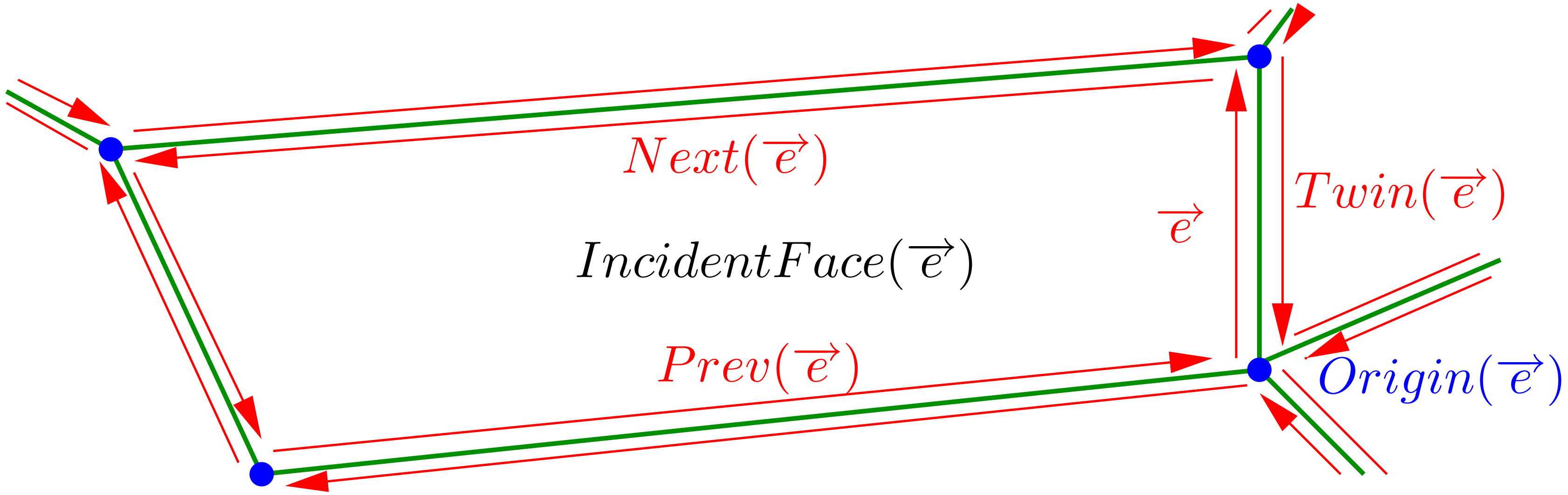


- not proven here

Planar Straight Line Graphs

Planar Straight Line Graphs

- planar graph with only straight line edges
- also called *planar subdivision*
- a data structure: *doubly connected edge list*
  - each edge is replaced by two directed half–edges

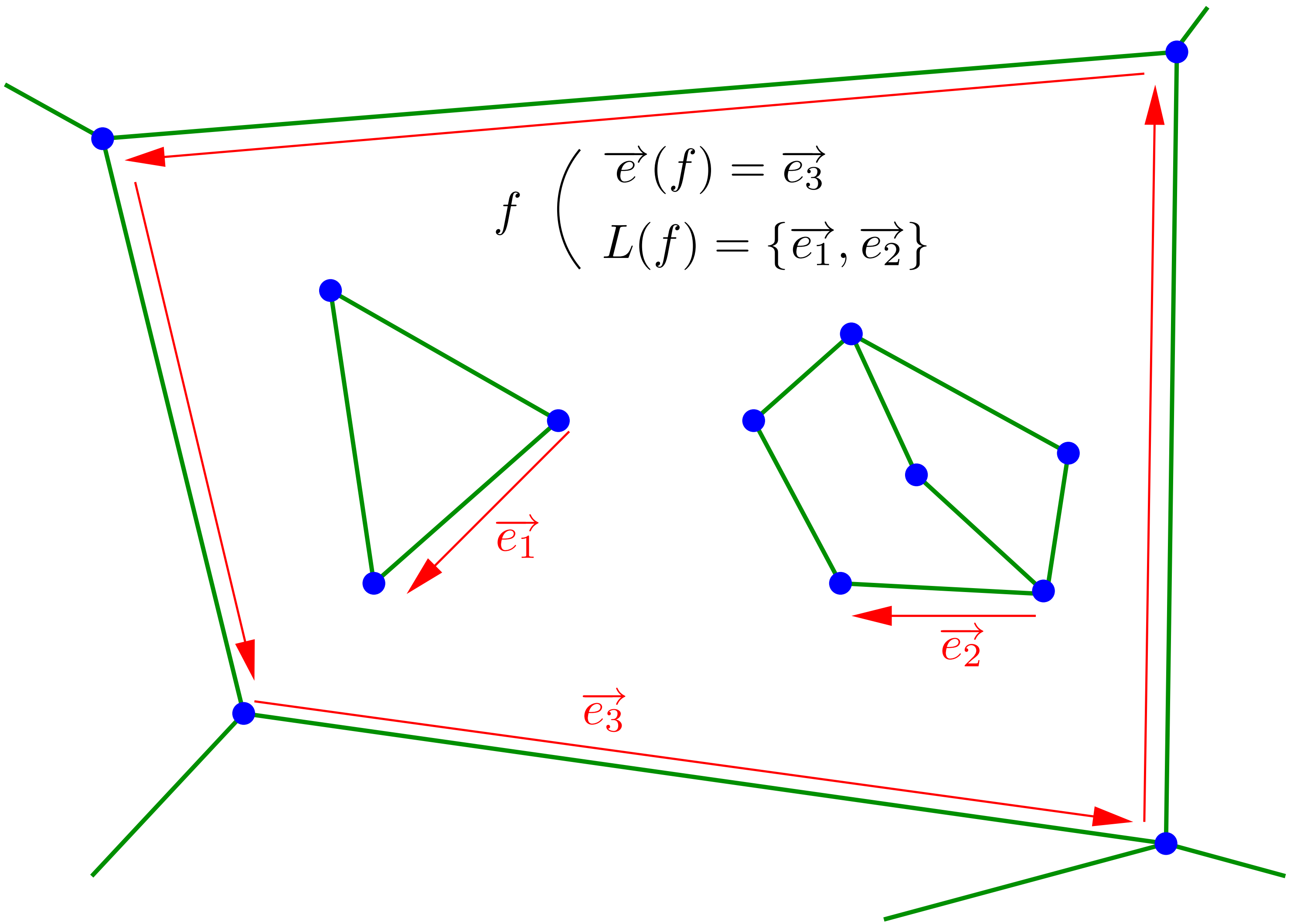


  - the half–edges enclosing a face form a counterclockwise cycle

Doubly connected edge list

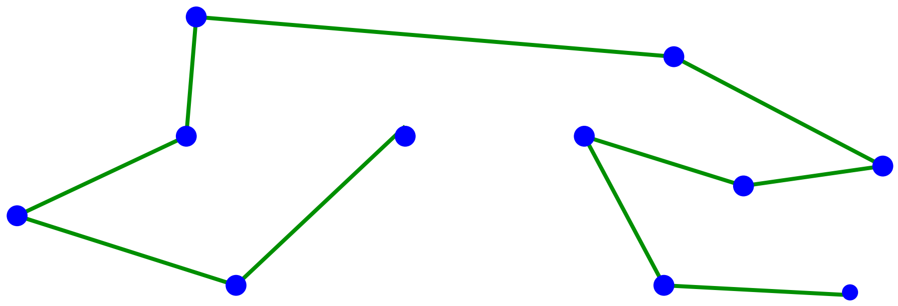
- vertex v
  - coordinates
  - an incident half–edge $IncidentEdge(v) = (v, w)$
- half edge $\overrightarrow{e}$
  - 3 edges $Twin(\overrightarrow{e})$, $Next(\overrightarrow{e})$, $Prev(\overrightarrow{e})$
  - vertex $Origin(\overrightarrow{e})$
  - a face $IncidentFace(\overrightarrow{e})$
- face f
  - a half–edge $\overrightarrow{e}(f)$ of its exterior boundary
  - a half–edge of each face contained in f; they are stored in a list $L(f)$

Faces in Doubly Connected Edge Lists

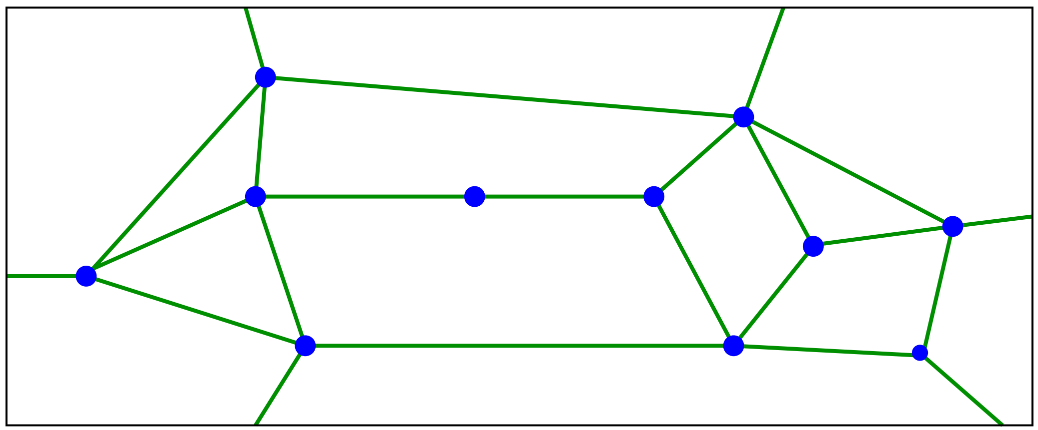


Special Cases

- Polyline: the edges form a chain



- Convex subdivision: all faces are convex

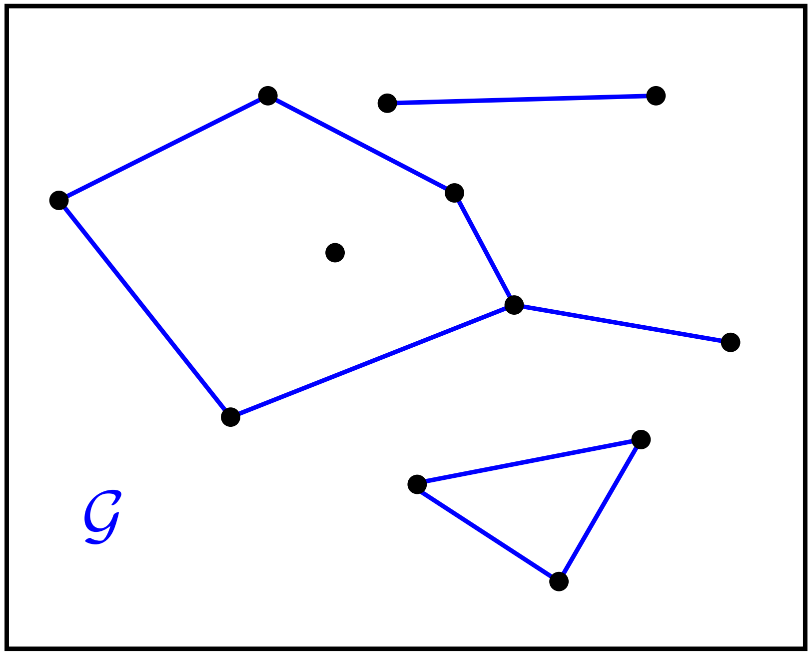


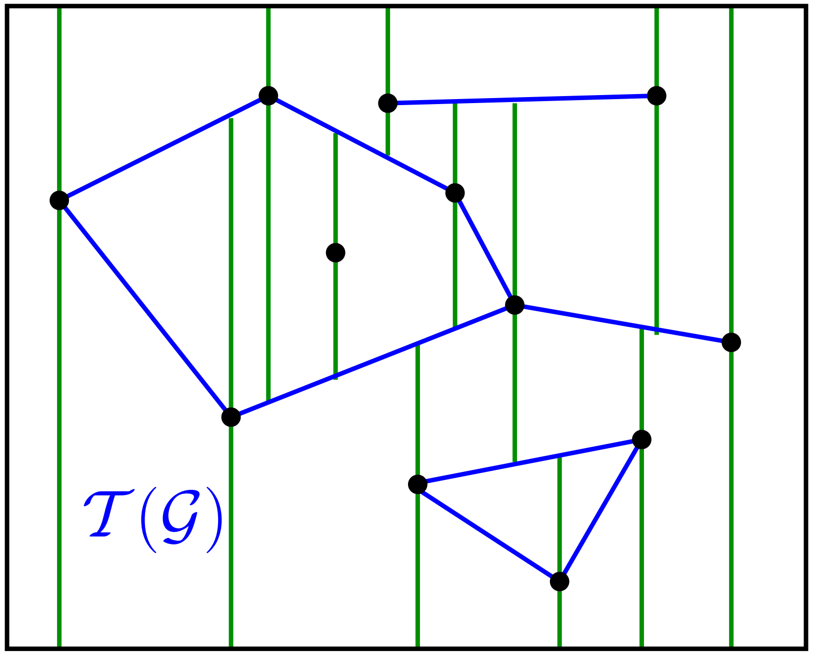
- polygons: a face of a PLSG (see below)

Trapezoidal map

Trapezoidal map

- start with a PSLG $\mathcal{G}$
- the trapezoidal map $\mathcal{T}(\mathcal{G})$ is the convex subdivision obtained by drawing vertical edges downward and upward from each vertex



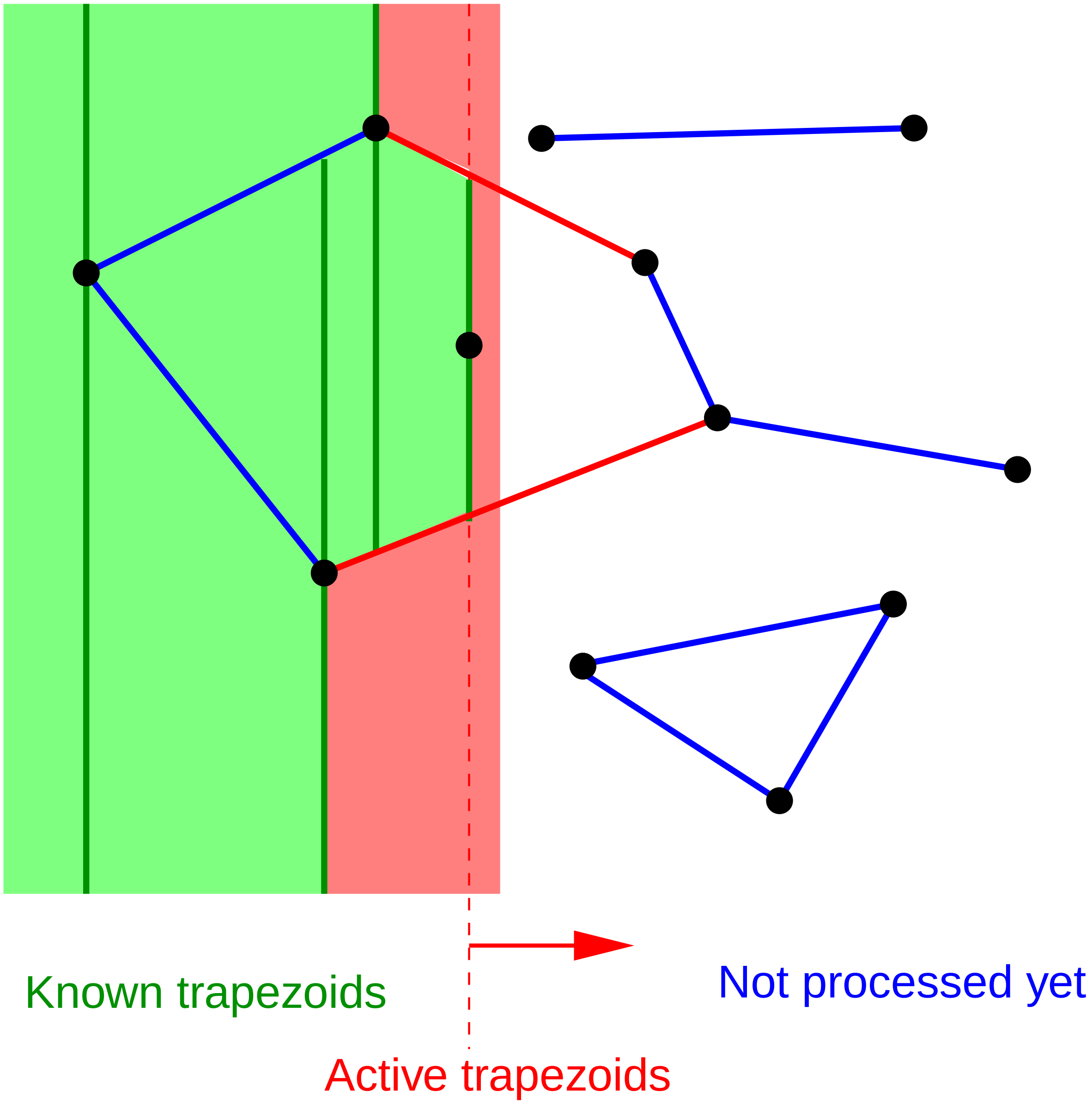


- we draw a bounding box around $\mathcal{G}$ so that there is no infinite face, hence all faces of $\mathcal{T}(\mathcal{G})$ trapezoids

Computing $\mathcal{T}(\mathcal{G})$

- assume $\mathcal{G}$ has n vertices
- input: a representation of $\mathcal{G}$ (for instance, a doubly connected edge list)
- output: a representation of $\mathcal{T}(\mathcal{G})$
- general position assumption: no two vertices have same x–coordinate
- idea: we will use plane sweep
- a modified version of the intersection detection algorithm
- first step: sort vertices by increasing x–coordinate
- an event: the sweep line reaches a vertex of $\mathcal{G}$

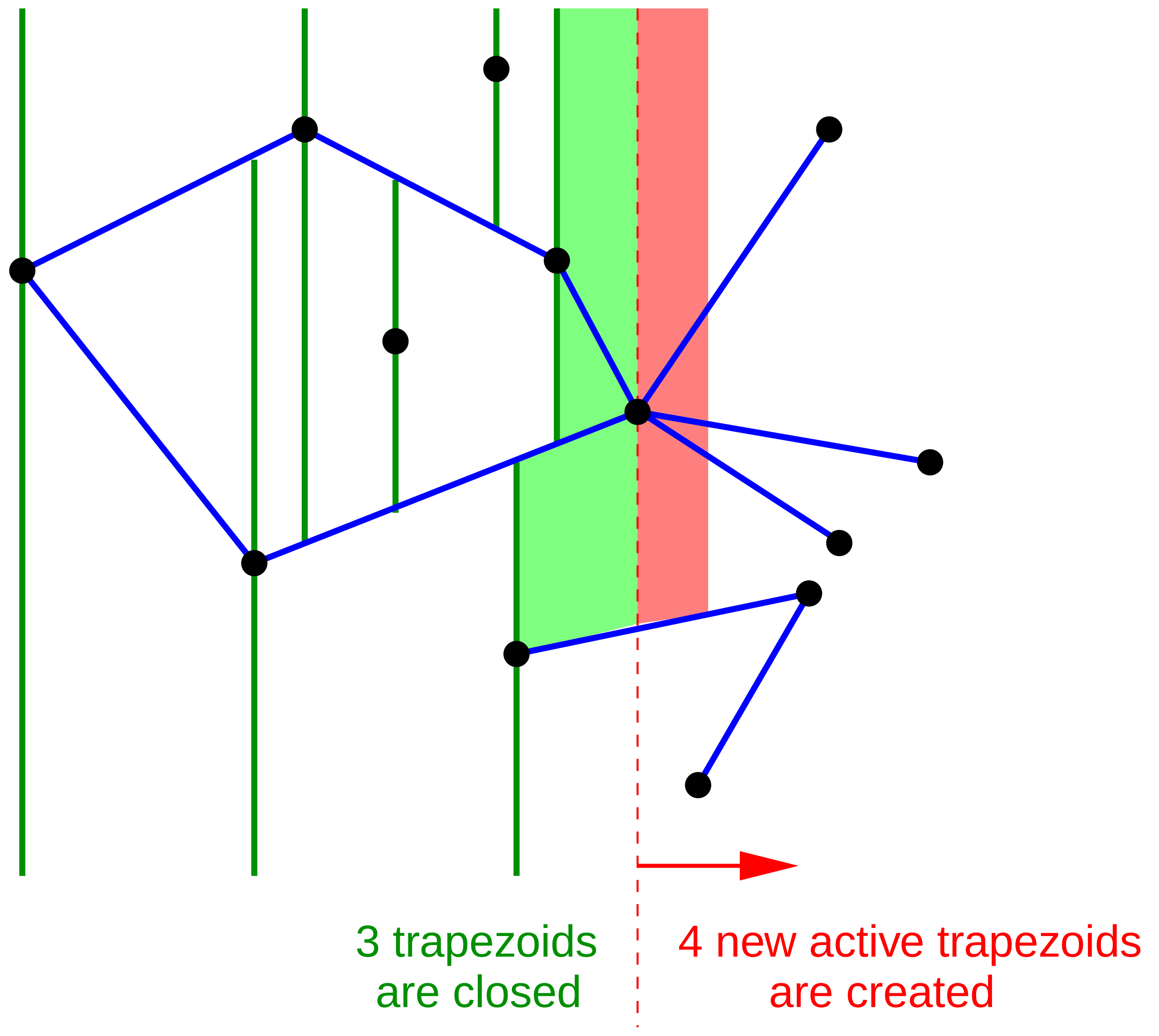
Computing $\mathcal{T}(\mathcal{G})$



Computing $\mathcal{T}(\mathcal{G})$

- invariants
  - we know the trapezoids that lie on the left of the sweep line
  - *active trapezoids*: trapezoids that intersect the sweep line
  - we know the order of the active trapezoids along the sweep line
  - we know the left, top and bottom edges of each active trapezoid
- an event: close some active trapezoids and create new ones

Computing $\mathcal{T}(\mathcal{G})$



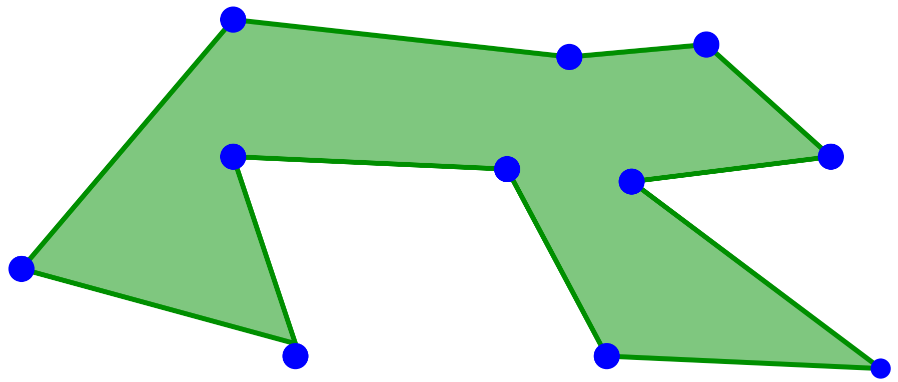
Computing $\mathcal{T}(\mathcal{G})$

- at event i suppose k_i trapezoids are closed or created
- event i can be handled in $O(k_i \log n)$ time
- amortized analysis
  - $\mathcal{T}(\mathcal{G})$ has at most $3n$ vertices
  - so there are $O(n)$ trapezoids
  - each trapezoid is created and closed one time only
  - so $\sum k_i = O(n)$
- overall, the algorithm runs in $O(n \log n)$ time

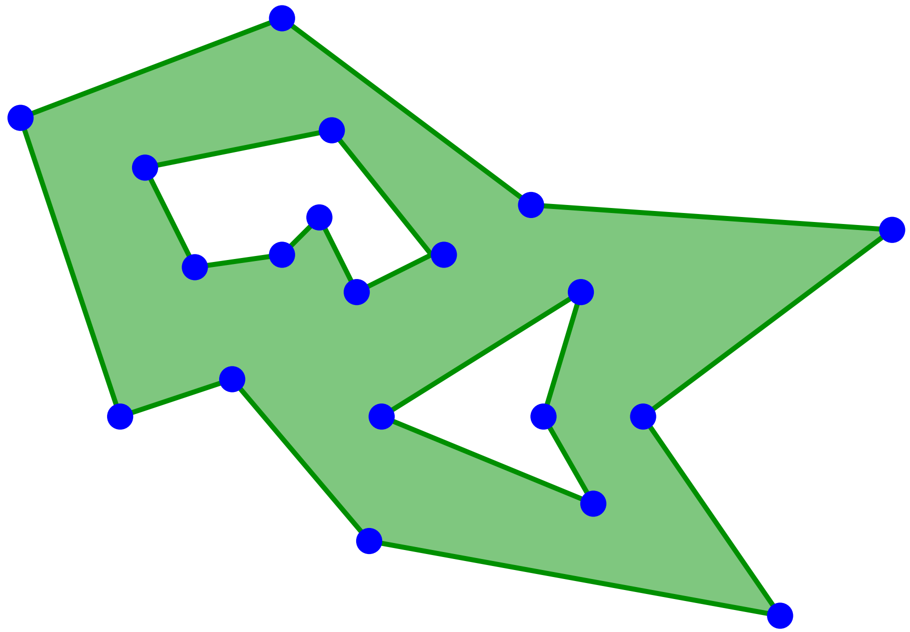
Polygons and Triangulations

Polygons

- A polygon is a face of a Planar Straight Line Graph
- A *simple polygon* is the region enclosed by a simple (=non–intersecting) polyline



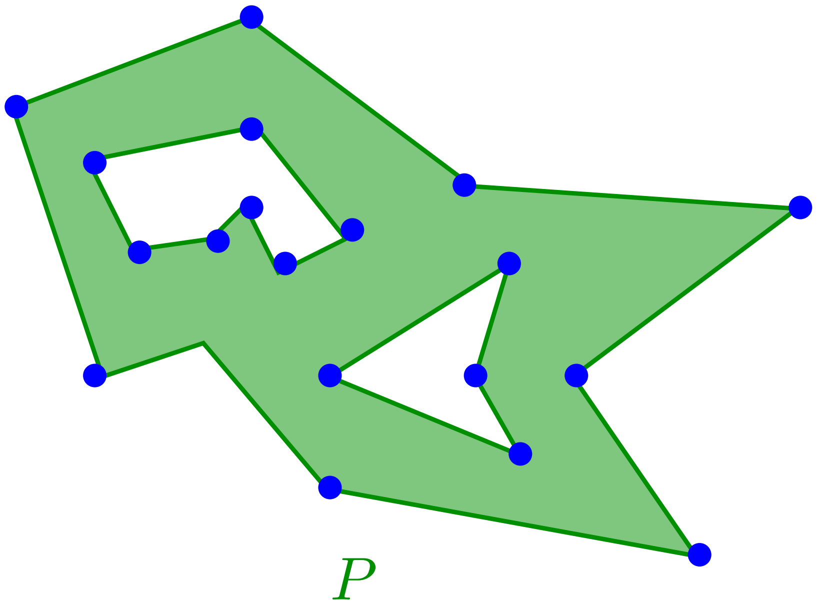
a simple polygon

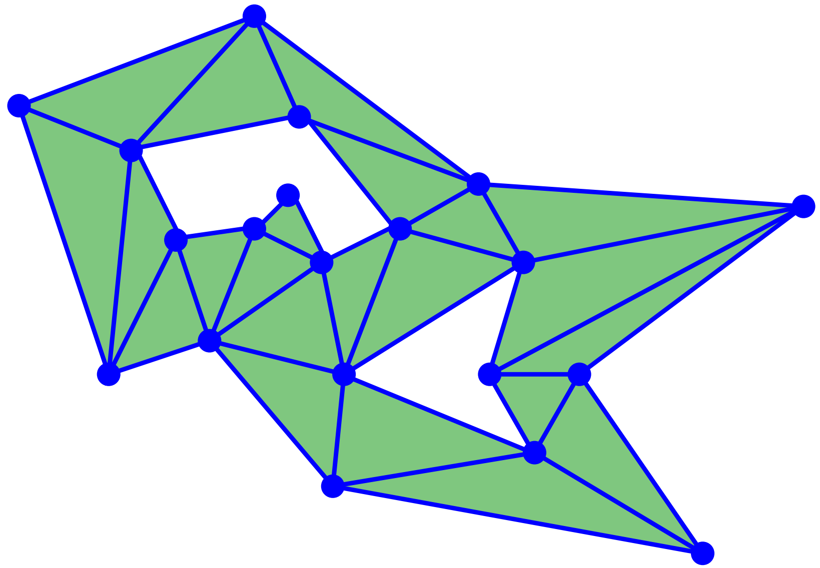


a polygon (with holes)

Triangulations

- A *Triangulation* of a polygon P is a partition of P into triangles whose vertices are the vertices of P





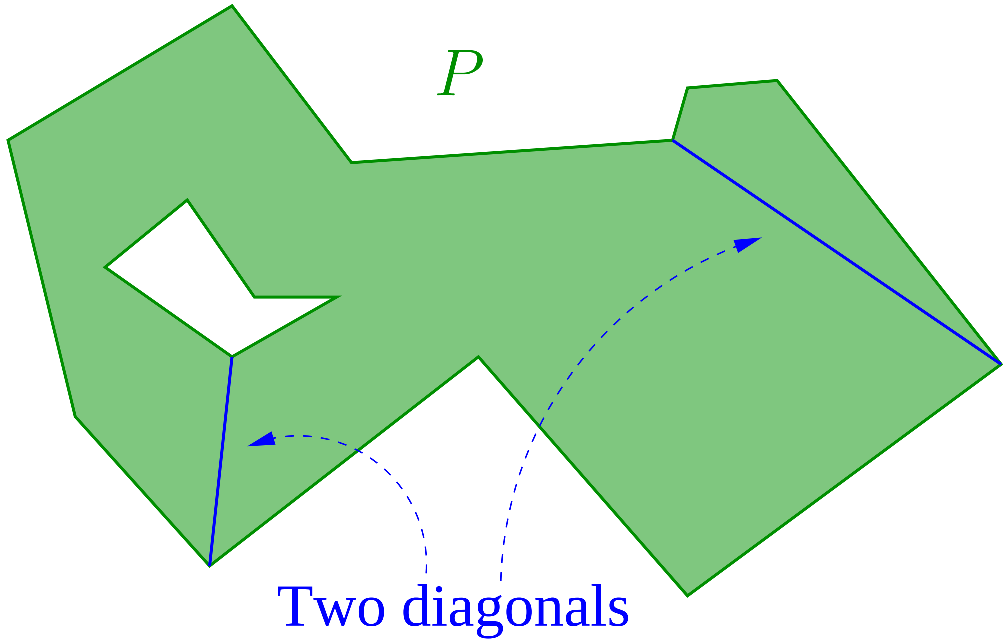
- A polygon may have several triangulations
- A triangulation is a planar straight line graph

Applications

- meshing $\Rightarrow$ scientific computing
- visibility problems
  - graphics
  - art gallery problem (see Notes page 27)
- preprocessing step of many geometric algorithms

Existence of a triangulation

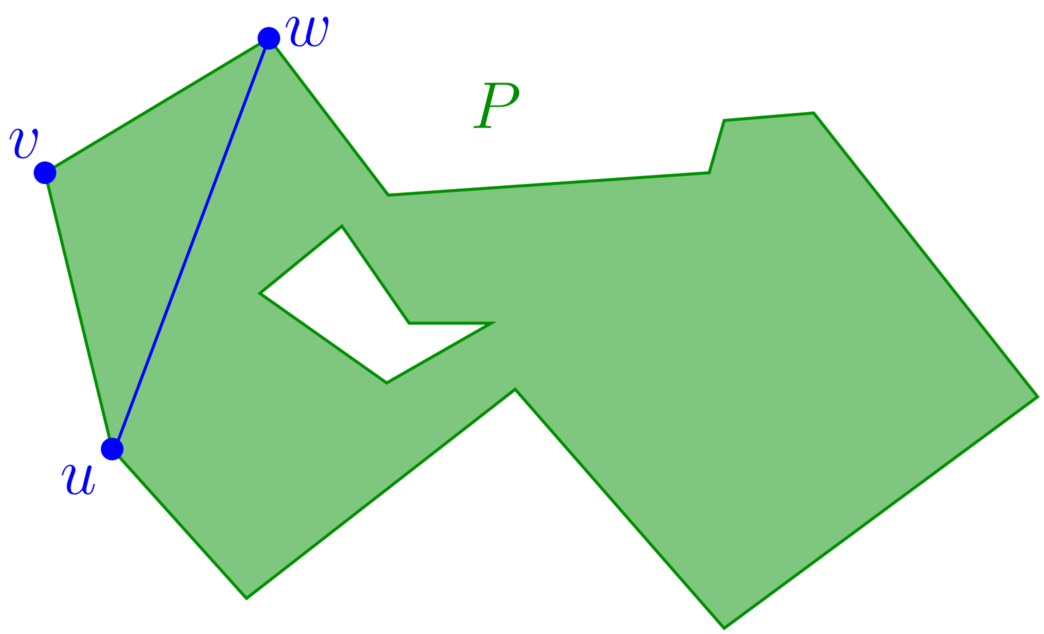
- We prove that every polygon P admits a triangulation
- definition: a *diagonal* of P is a line segment $\overline{pq}$ such that p and q are vertices of P and the interior of $\overline{pq}$ is in the interior of P.



- Lemma 1: every polygon P with more than three vertices admits a diagonal

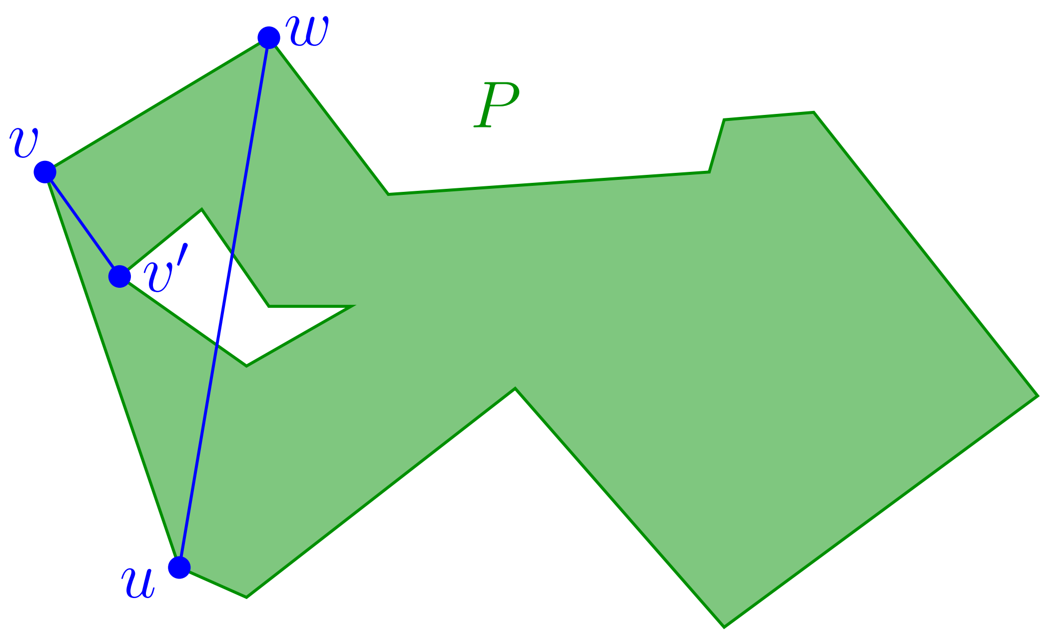
Proof of Lemma 1

- let v be the leftmost vertex of P
- let u and w be its neighbors
- if $\overline{uw}$ is a diagonal we are done



Proof of Lemma 1

- if $\overline{uw}$ is not a diagonal
- let v' be the vertex in triangle (u, v, w) that is farthest from $\overline{uw}$



- then $\overline{vv'}$ is a diagonal: if an edge was crossing it, one of its endpoints would be farther from $\overline{uw}$ and inside (u, v, w)

Proof of existence

- Theorem: any polygon P admits a triangulation
- Proof:
  - if P has 3 vertices, then P is its own triangulation
  - otherwise insert a diagonal of f
    - if P becomes disconnected, we know by induction that the two faces can be triangulated, so we are done
    - if P is still connected, repeat the process of inserting a diagonal
  - this algorithm halts since $|E| < |V|^2$ and $|V|$ is constant

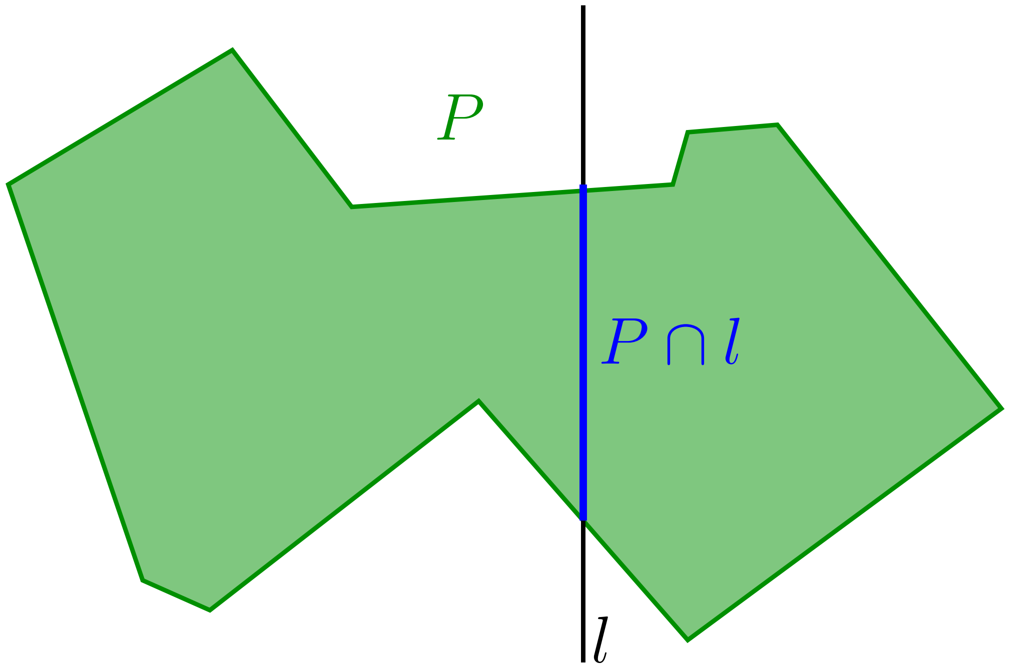
More results

- any triangulation of a simple polygon with n vertices has $n-2$ faces and $n-3$ diagonals
- we can find a diagonal in $O(n)$ time
- we can find a triangulation in $O(n^2)$ time
- is there a faster algorithm?
  - yes, there is an optimal $O(n \log n)$ time and $O(n)$ space algorithm
  - this is what we will see next
- there is an $O(n)$ time algorithm for simple polygons
  - very difficult, we do not study it

Triangulating a monotone polygon

Definition

- an x–monotone polygon is a polygon such that for all vertical line l, the intersection $P \cap l$ is a line segment

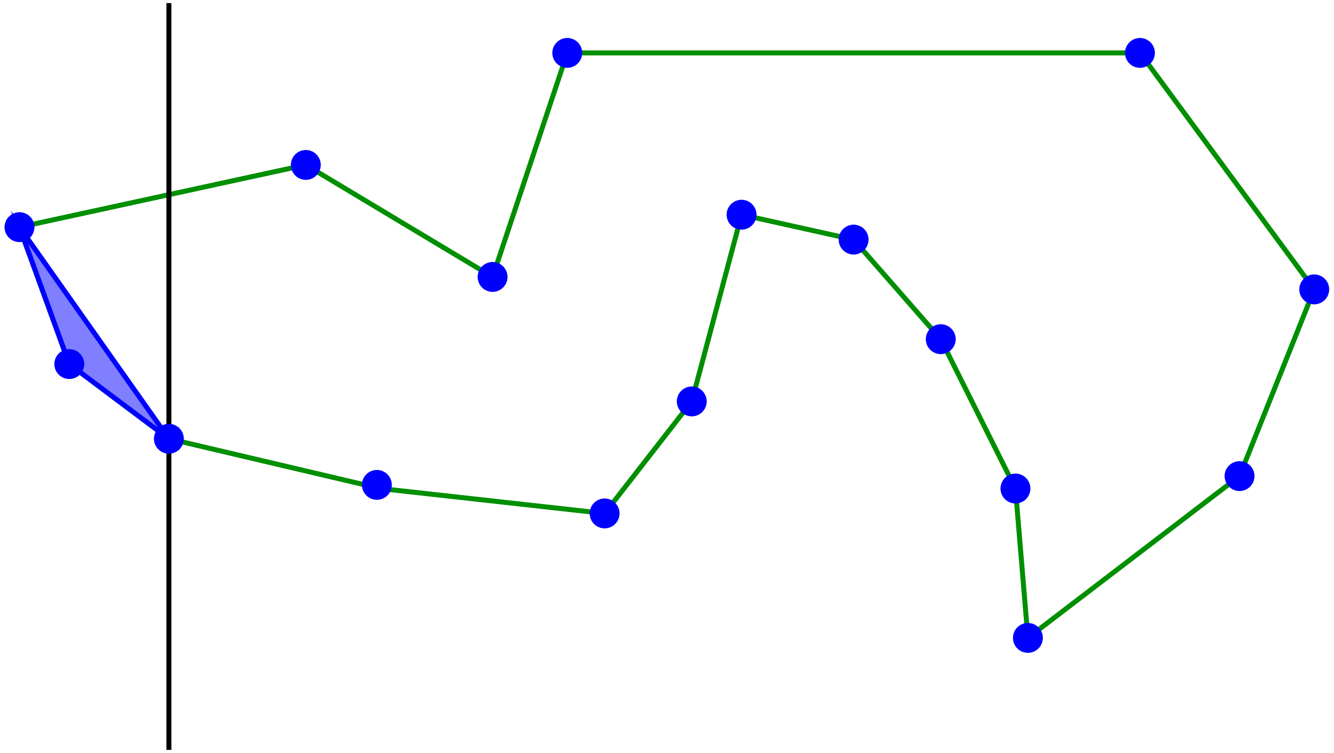


- equivalently, it is a simple polygon whose boundary consists of two x–monotone polylines

Algorithm

- plane sweep approach
- the sweep line l moves from left to right and stops at each vertex of P
  - we can sort these vertices in $O(n \log n)$ time
  - we can also do it in $O(n)$ time. How?

Example



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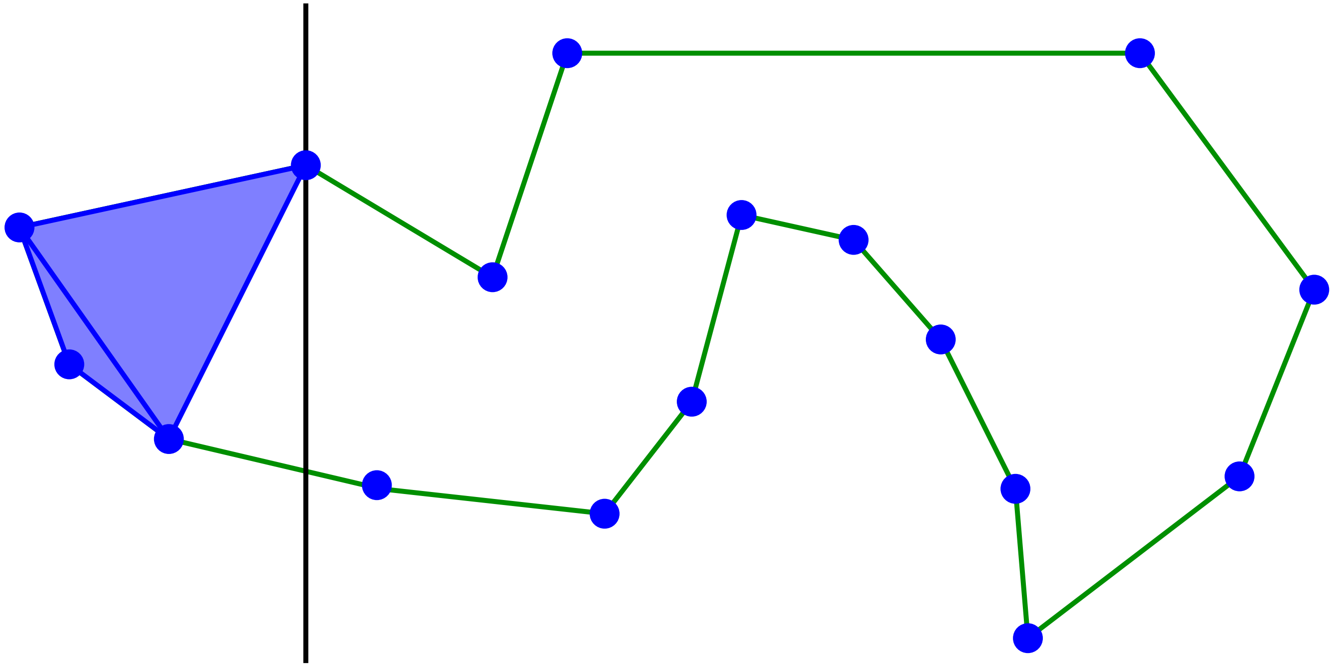

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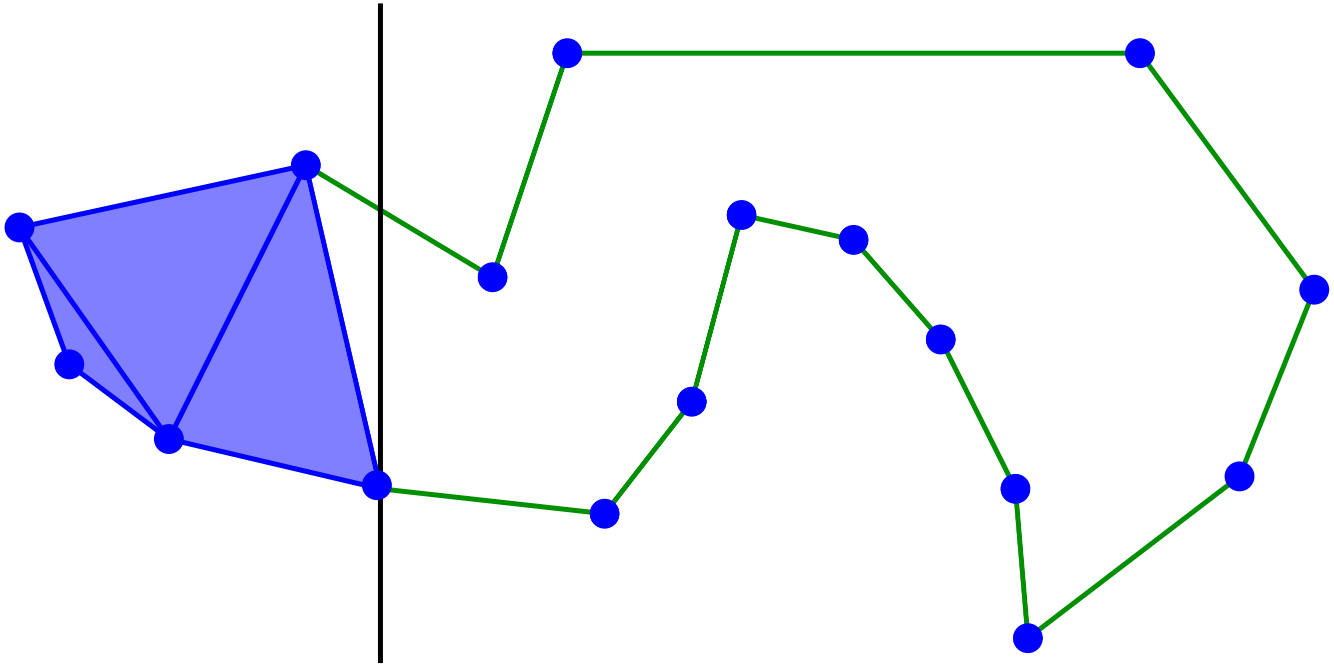

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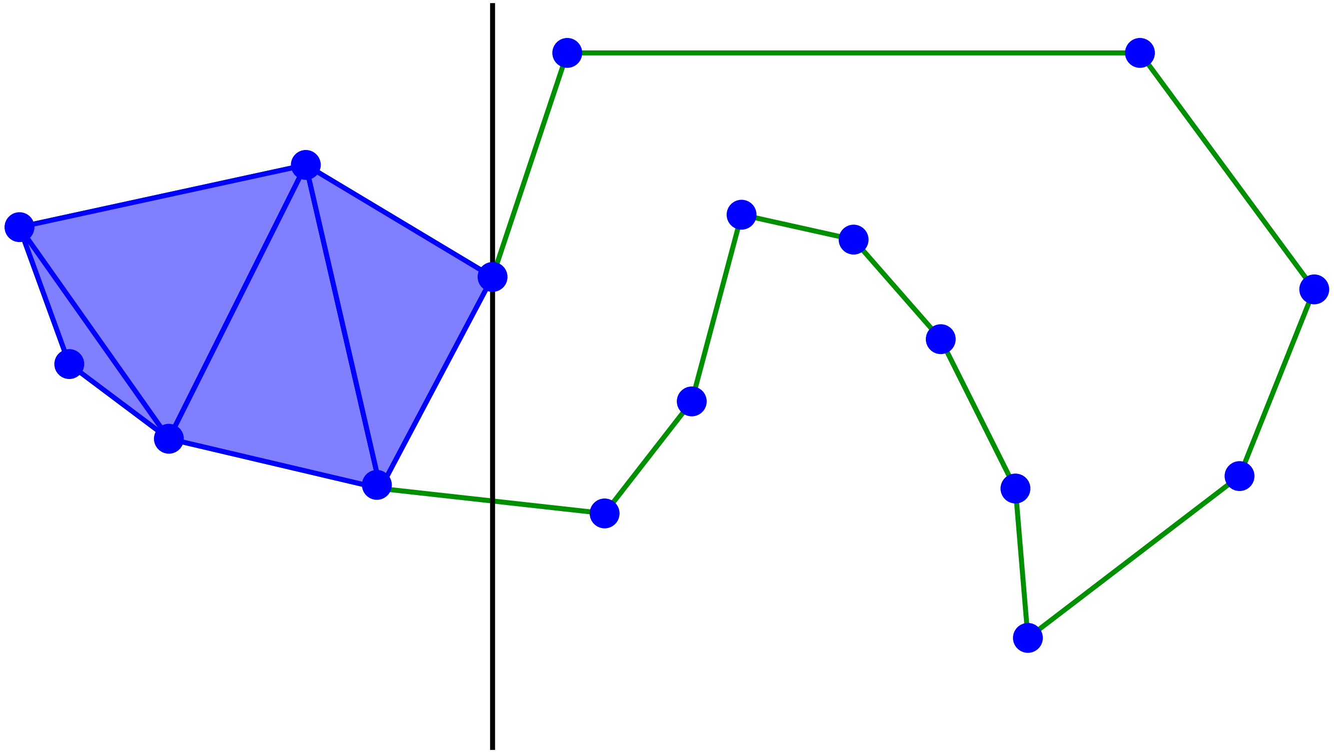

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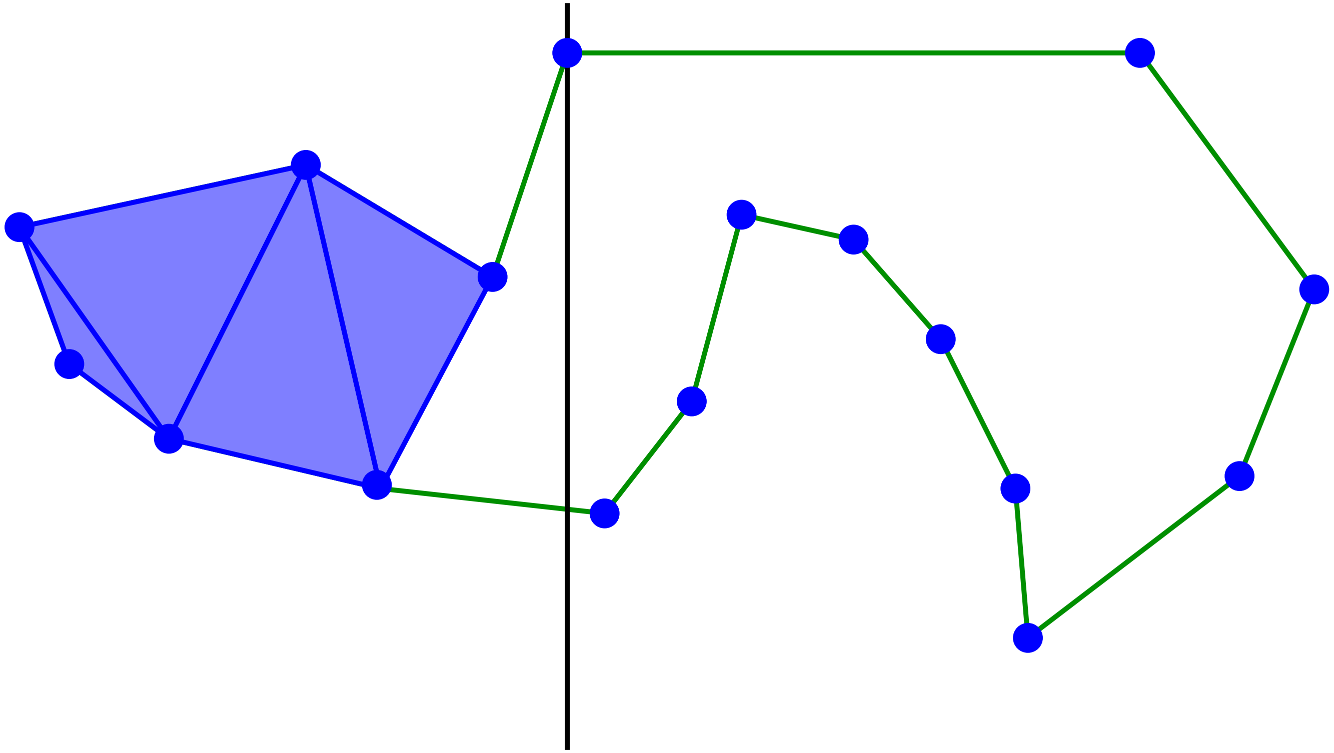

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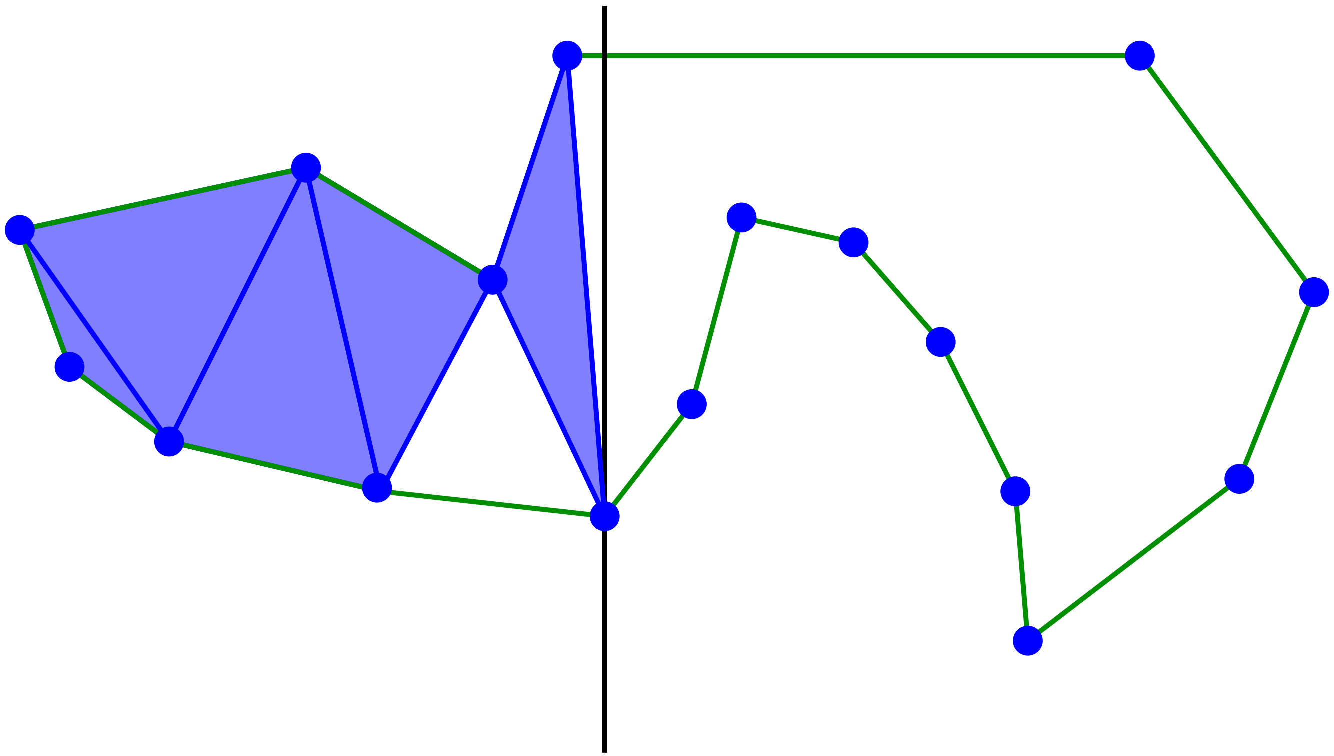

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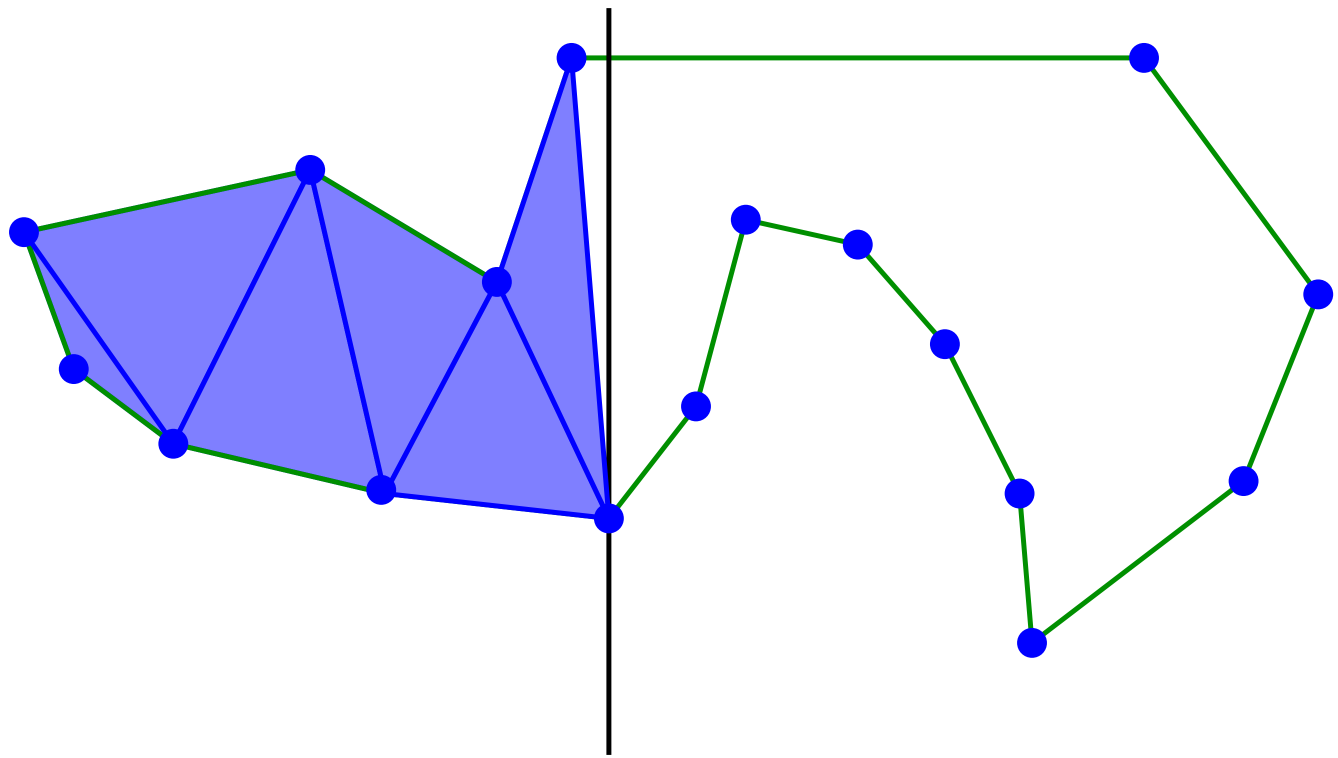

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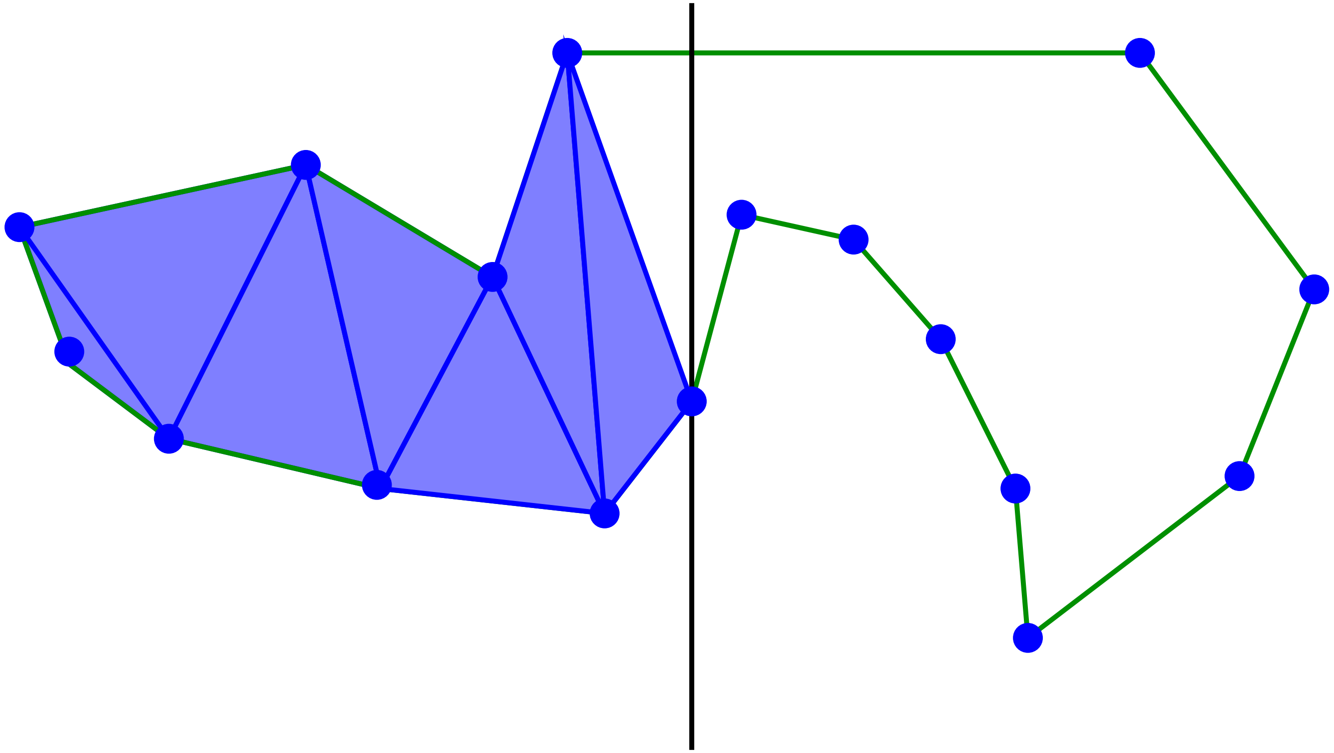

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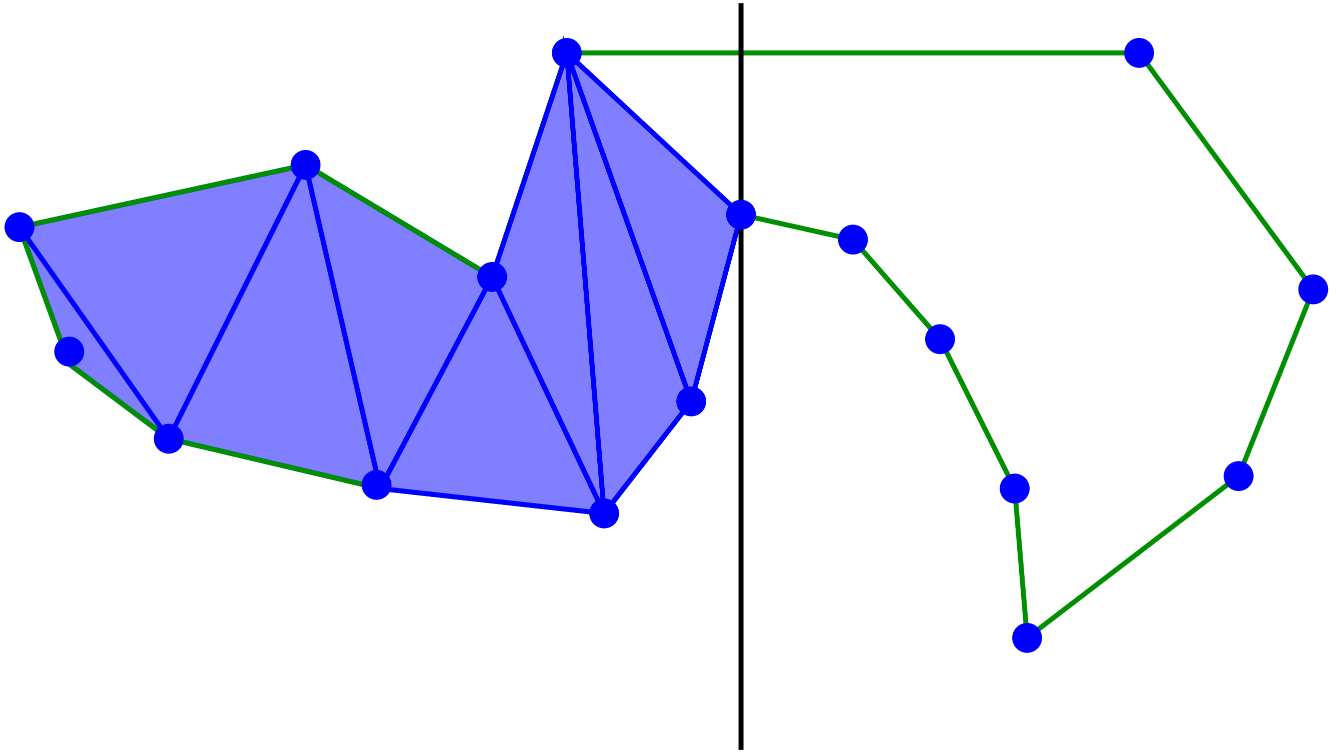

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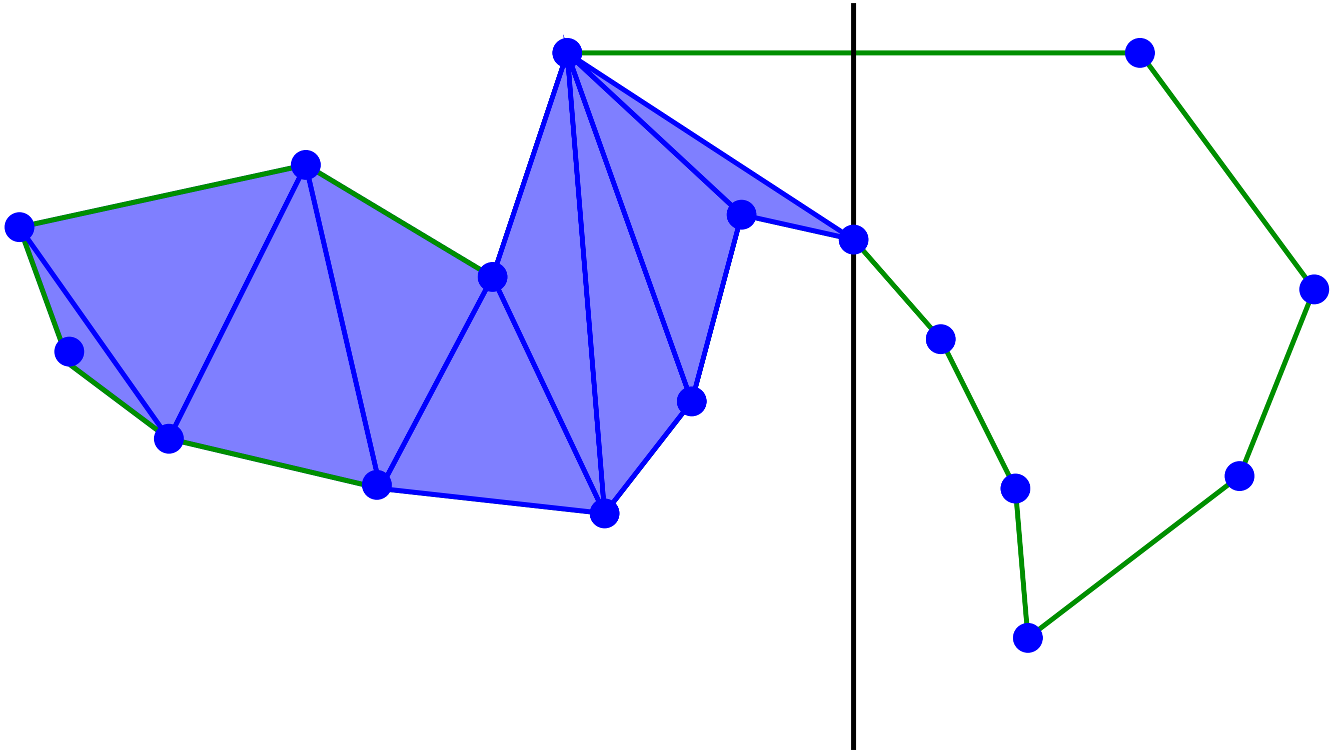

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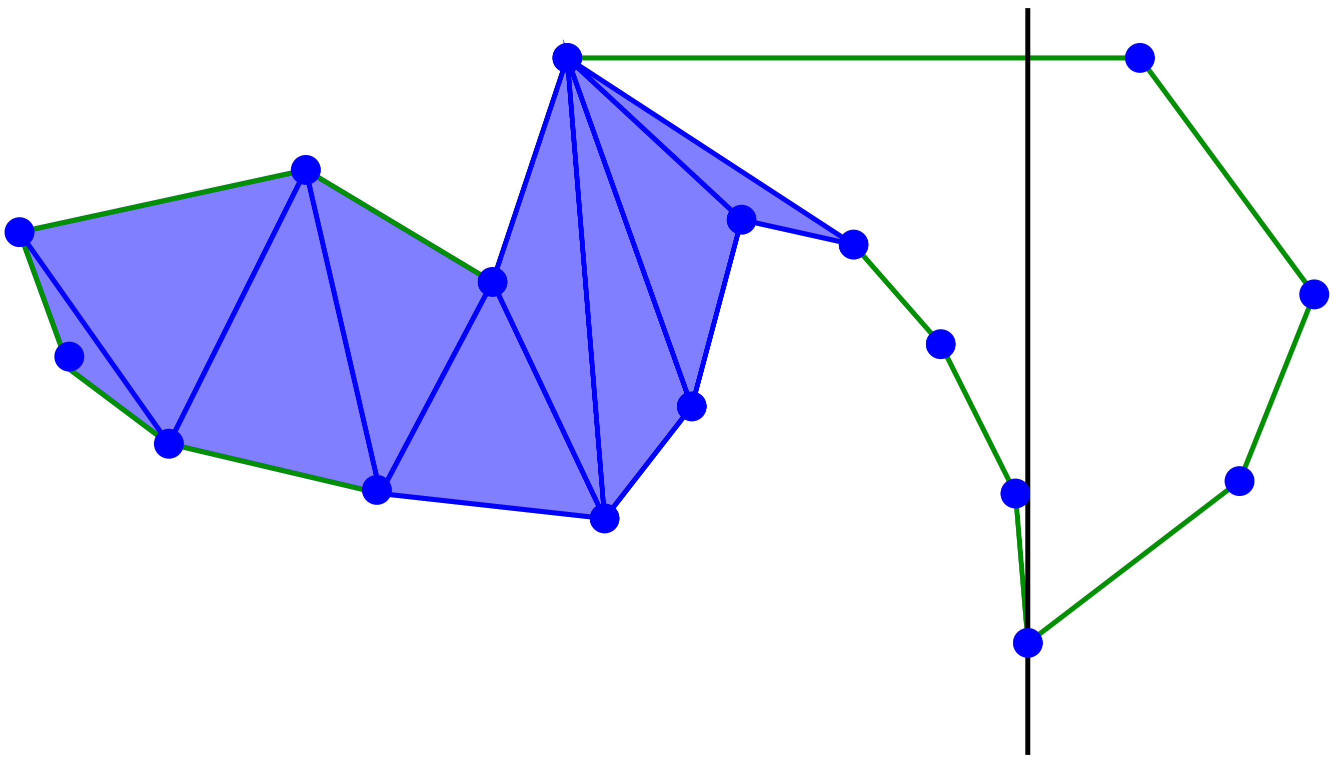

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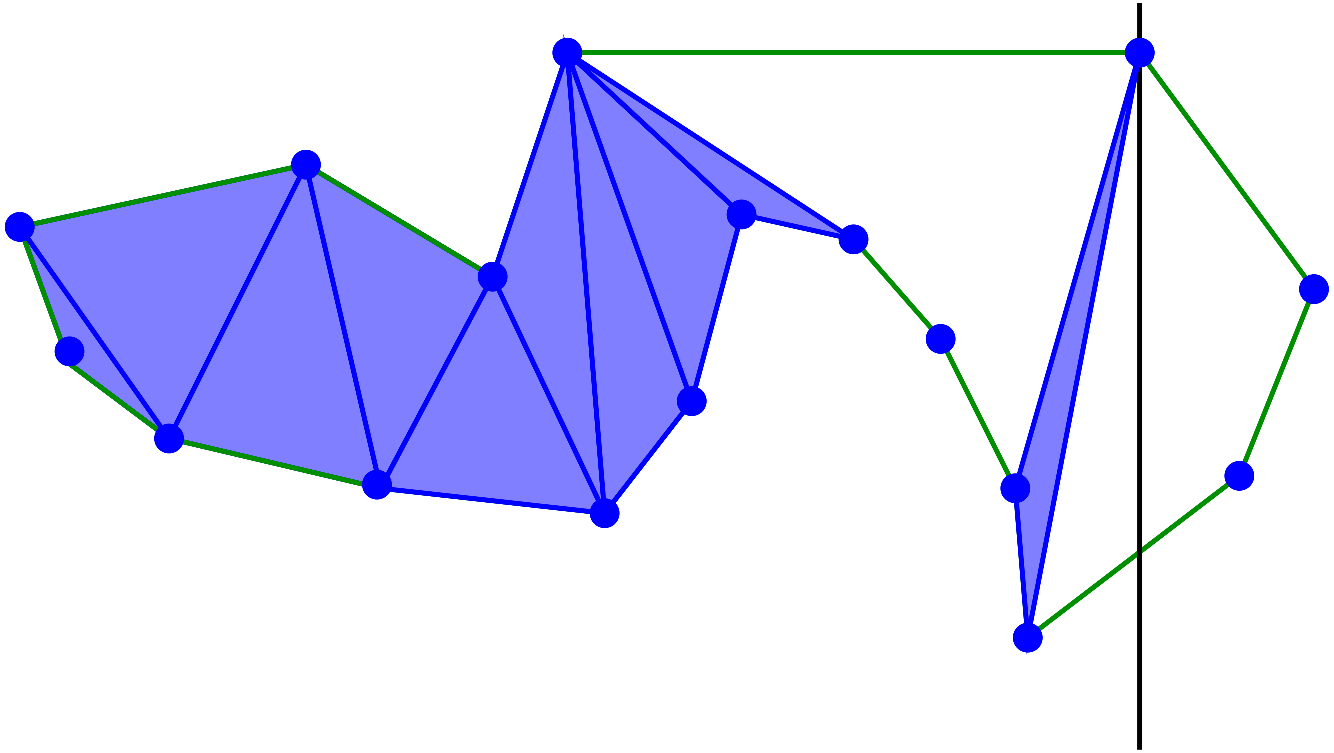

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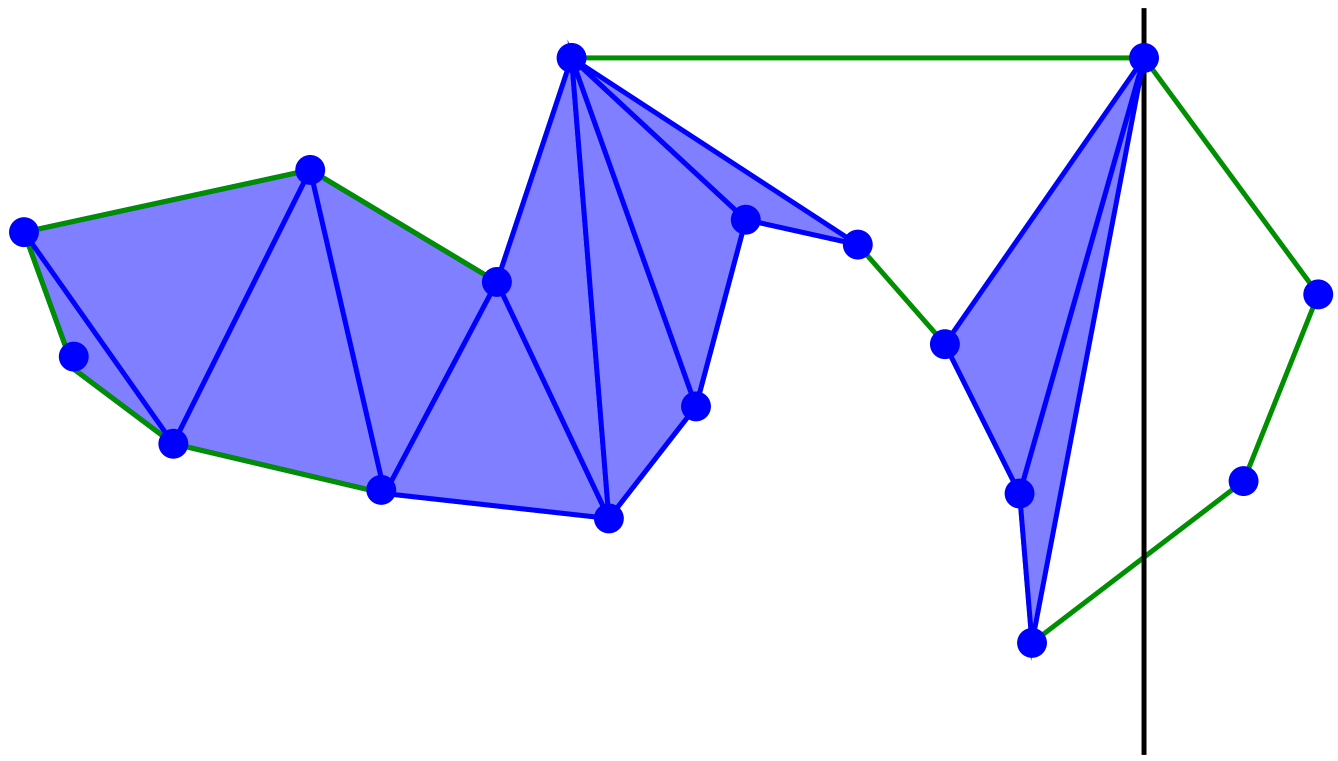

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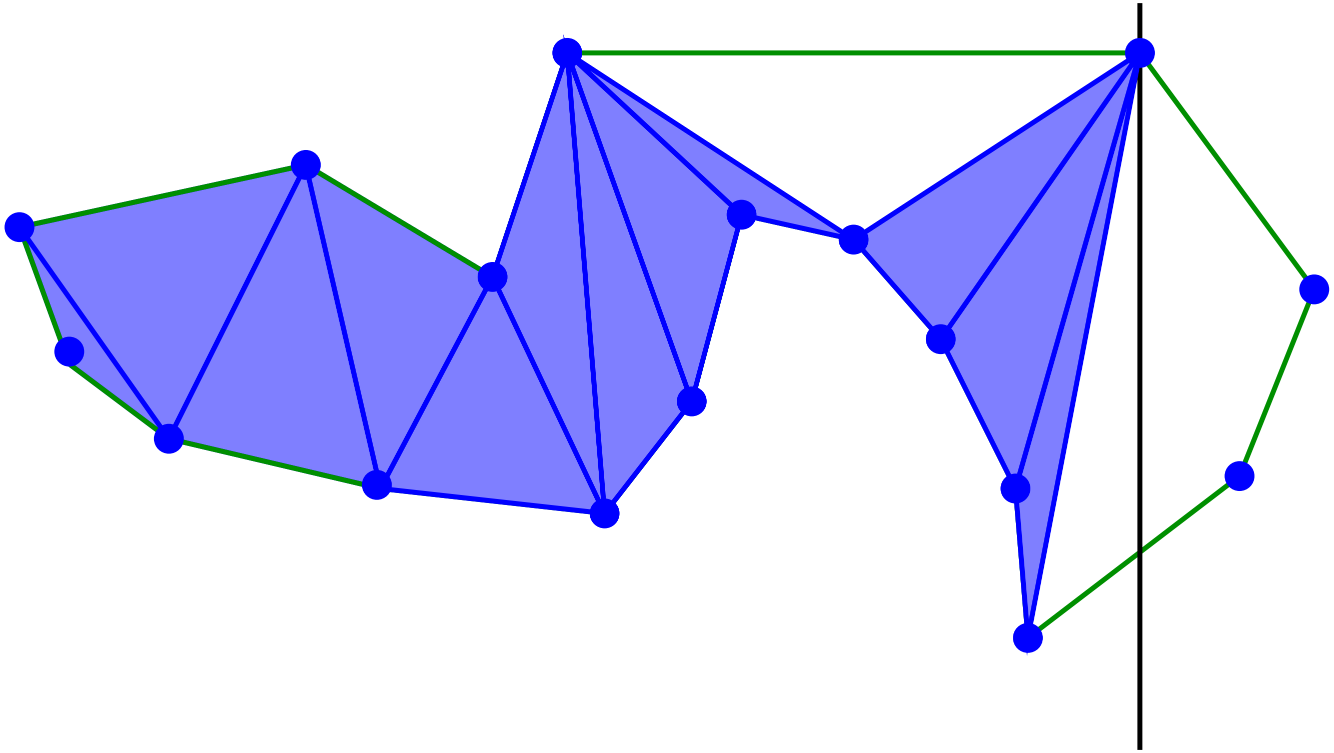

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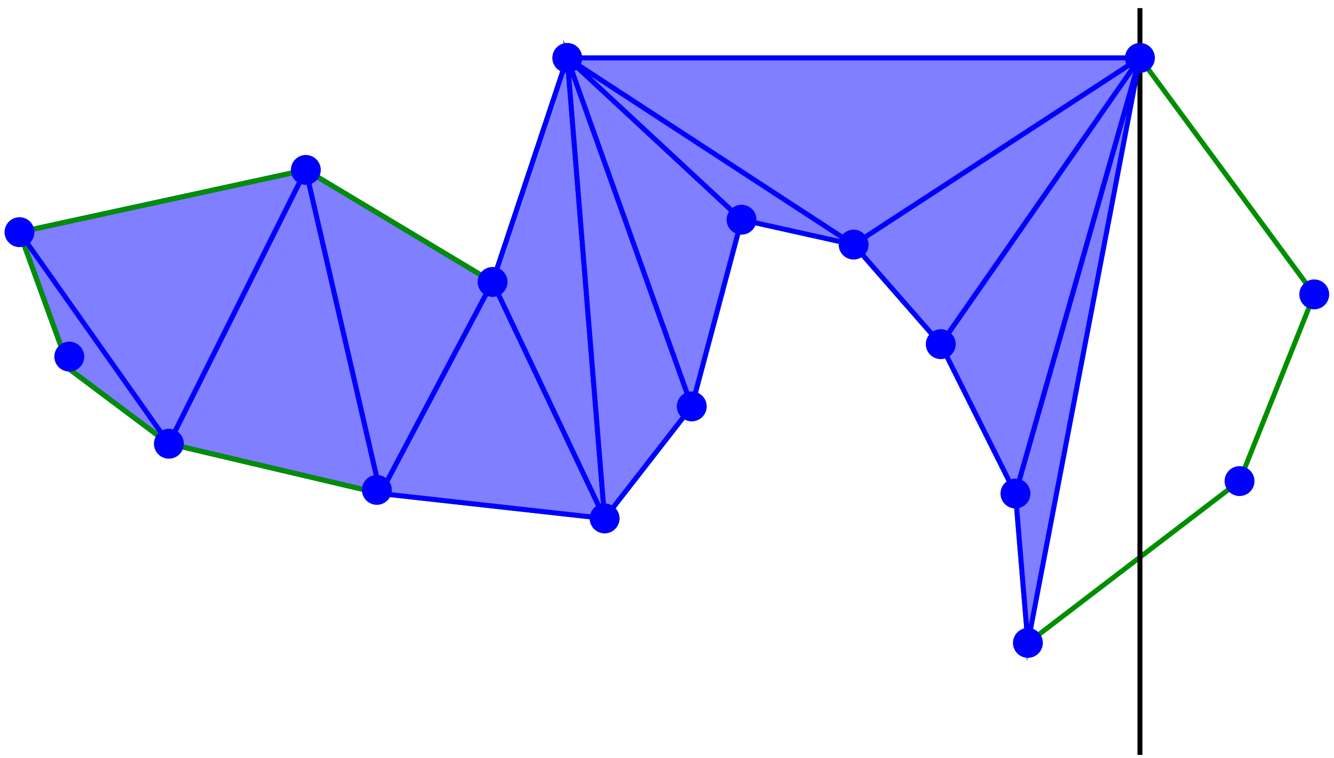

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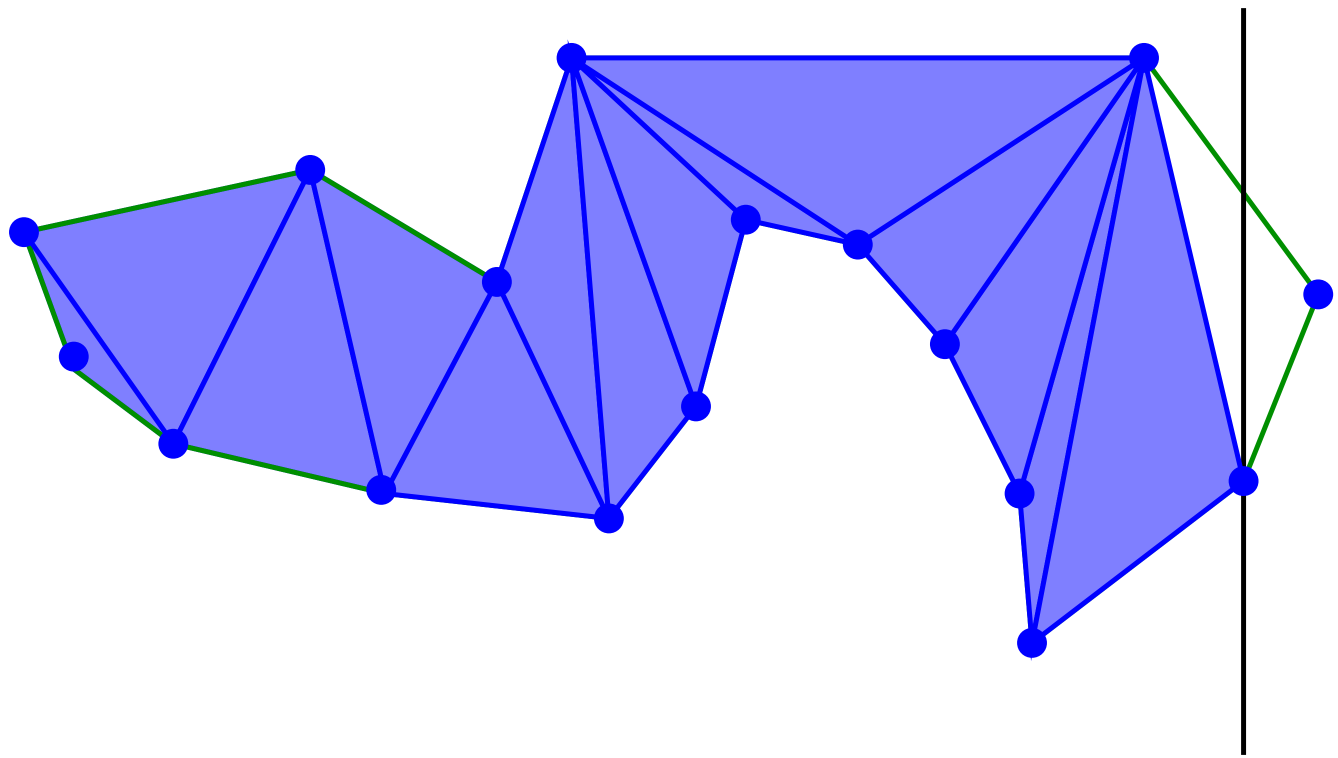

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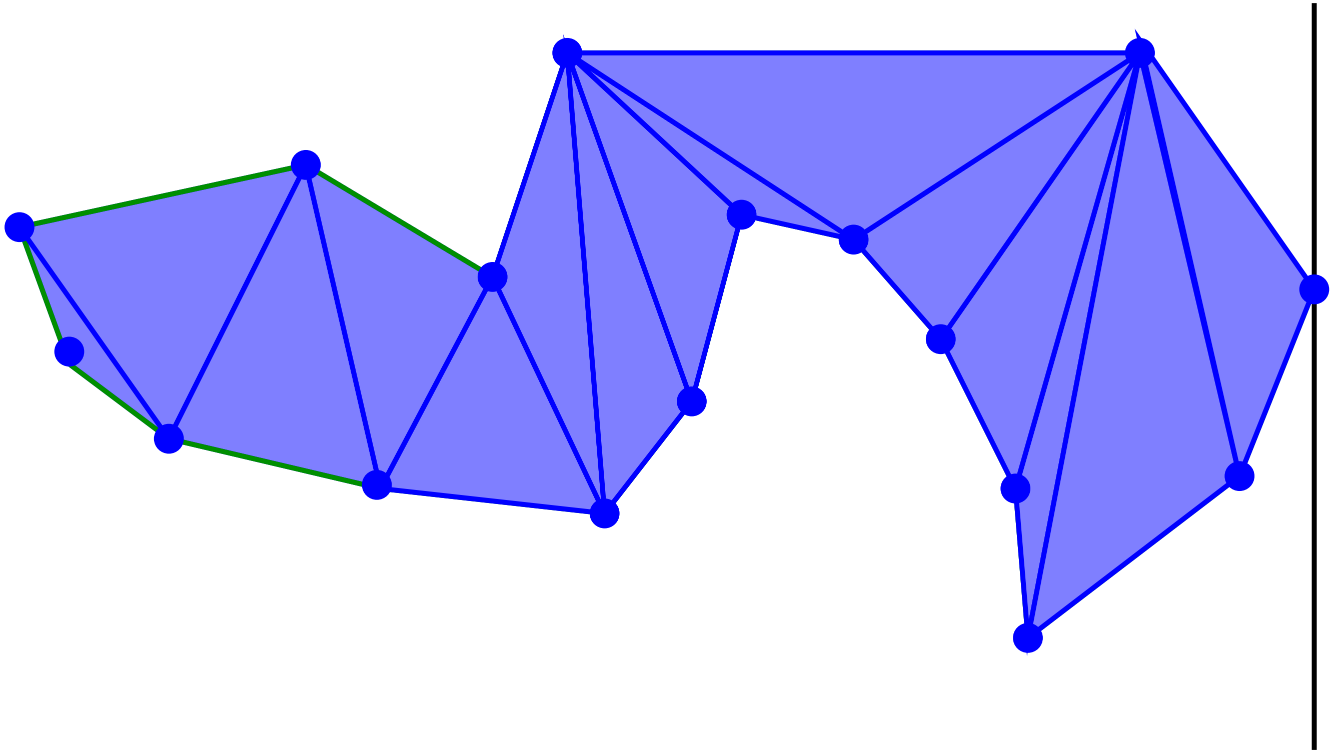
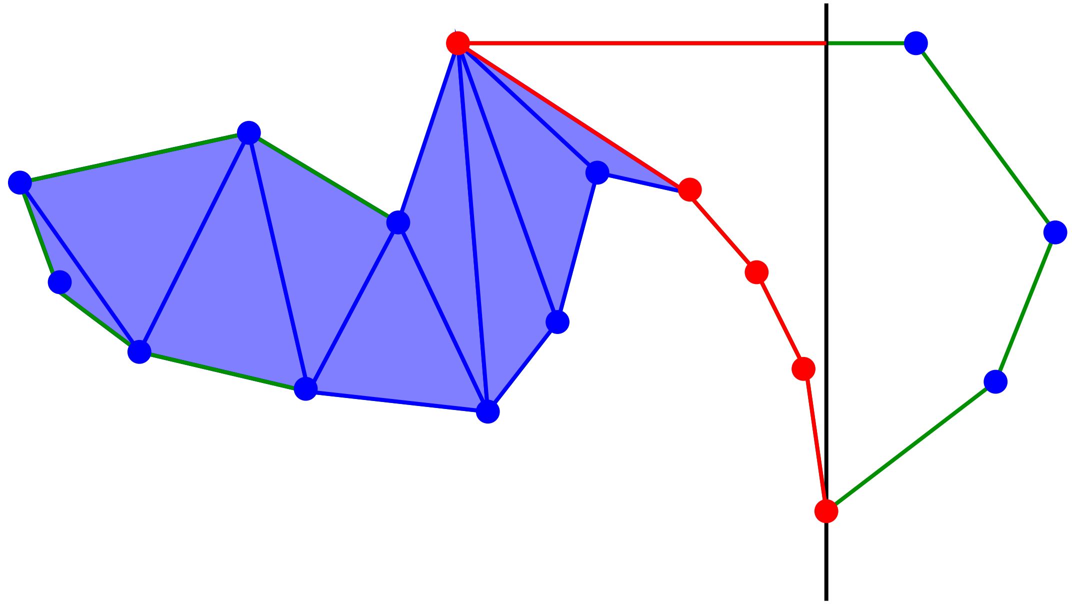


Proof of correctness

- Invariant:
  - the non triangulated region to the left of the sweep line is delimited by an edge on one side and a reflex chain on the other side



  - we can maintain this invariant (see D. Mount notes)

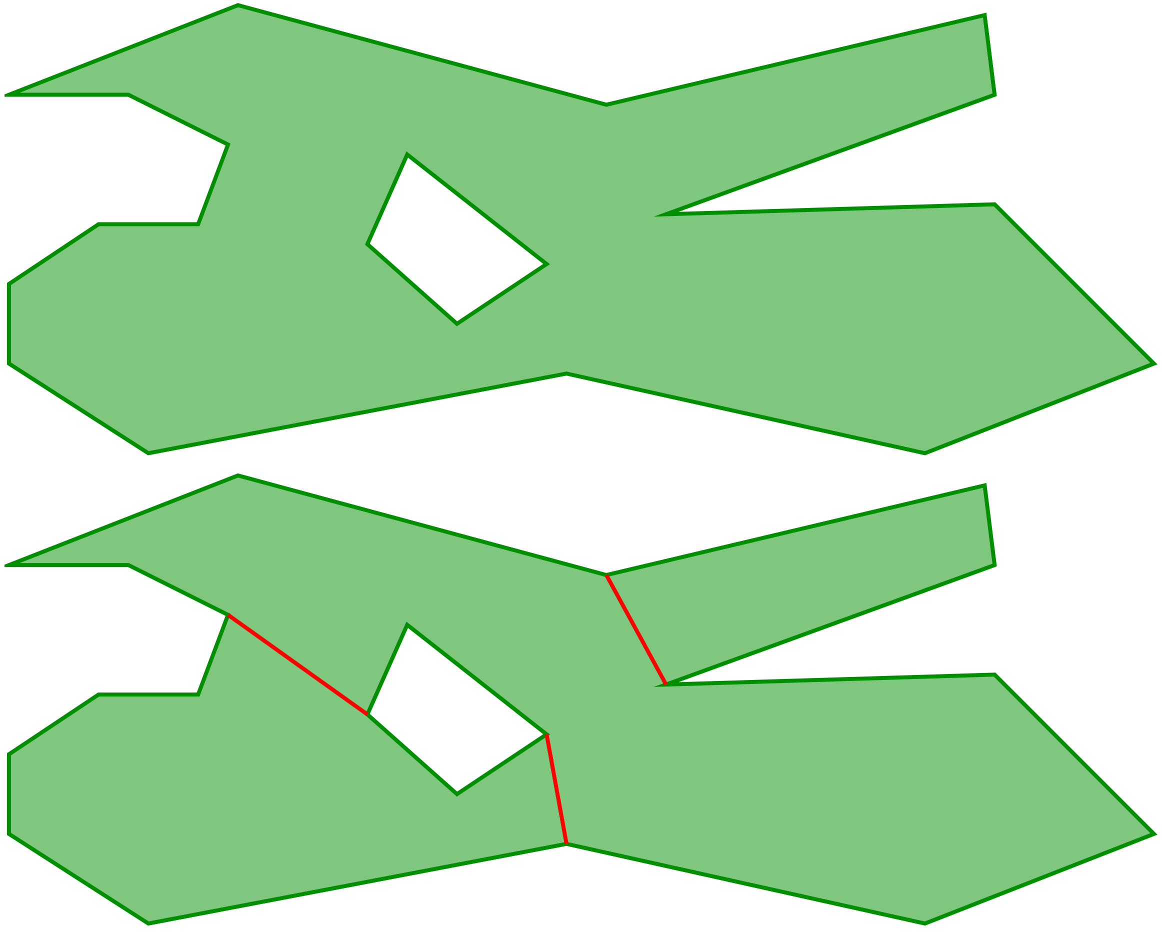
Analysis

- vertices can be sorted along the x–axis in $O(n)$ time
- we maintain the reflex chain in a stack
  - push and pop in $O(1)$ time
- each vertex is pushed and popped at most once
- this algorithm runs in optimal $\Theta(n)$ time
- we can use Doubly Connected Edge Lists

Partitioning a polygon into monotone pieces

Problem

- we want to partition a polygon P into a collection of x–monotone polygons with same vertex set



Algorithm

- we will find an $O(n \log n)$ time algorithm
- combined with previous section, it yields an $O(n \log n)$ time algorithm to triangulate an arbitrary polygon
- idea: first compute the trapezoidal map
  - it takes $O(n \log n)$ time
  - exercise for next week: how to obtain a monotone partition once we have the trapezoidal map?