Flip-Flop: Convex Hull Construction via Star-Shaped Polyhedron in 3D*

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Abstract

Flipping is a local and efficient operation to construct the convex hull in an incremental fashion. However, it is known that the traditional flip algorithm is not able to compute the convex hull when applied to a polyhedron in \( \mathbb{R}^3 \). Our novel Flip-Flop algorithm is a variant of the flip algorithm. It overcomes the deficiency of the traditional one to always compute the convex hull of a given star-shaped polyhedron with provable correctness. Applying this to construct convex hull of a point set in \( \mathbb{R}^2 \), we develop ffHull, a flip algorithm that allows nonrestrictive insertion of many vertices before any flipping of edges. This is unlike the well-known incremental fashion of strictly alternating between inserting a single vertex and flipping. The new approach is not only simpler and more efficient for CPU implementation but also maps well to the massively parallel nature of the modern GPU. As shown in our experiments, ffHull running on the CPU is as fast as the best-known convex hull implementation, qHull. As for the GPU, ffHull also outperforms all known prior work. From this, we further obtain the first known solution to computing the 2D regular triangulation on the GPU.


Keywords: Lawson’s flip, flipping, regular triangulation, Delaunay triangulation, incremental insertion, GPGPU

1 Introduction

Flipping is a local operation to transform one triangulation to another. In particular, any triangulation of a set of points \( S \) in \( \mathbb{R}^2 \) can be transformed into the Delaunay triangulation (DT) of \( S \) by Lawson’s flip algorithm [1977]. This algorithm repeatedly flips any edge in the triangulation that is not locally Delaunay till none exists, after which the result is the DT. The flipping operation can also be generalized to higher dimensions, but unlike the case in \( \mathbb{R}^2 \), Joe [1989] shows that in \( \mathbb{R}^3 \) the flip algorithm can be stuck at a local optimum. At that stage, there still are non-locally Delaunay facets but flipping them creates self-intersection. Edelsbrunner and Shah [1992] demonstrate the same situation when flipping on a regular triangulation (a generalization of the DT with weights on vertices) in \( \mathbb{R}^3 \). Figure 1(a) shows the example by Edelsbrunner and Shah where the weights of the vertices of the small triangle in the center are set so that the dashed edges are not locally regular, but flipping them creates self-intersection. To avoid being stuck at a local optimum, Joe [1991] proposes to insert points one by one followed by flipping to get to the DT after each insertion. That work is later extended to higher dimensions by Rajan [1991] and Edelsbrunner and Shah [1992].

The flipping operation can also be generalized to flipping on a polyhedron. In fact, by lifting the points in \( \mathbb{R}^n \) into \( \mathbb{R}^{n+1} \), the problem of constructing the regular triangulation in \( \mathbb{R}^n \) becomes the problem of constructing the lower hull in \( \mathbb{R}^{n+1} \) [Edelsbrunner and Seidel 1986]. Just for \( \mathbb{R}^3 \), to transform a polyhedron into the convex hull of its vertices, one repeatedly flips reflex edges to increase its volume. If self-intersection needs to be avoided after each flip, it is known that the flip algorithm also does not always terminate at the required convex hull [Aichholzer et al. 2002]. This is true even when the polyhedron is star-shaped: see Figure 1(b). In this example, the upper part of the star-shaped polyhedron is constructed similar to Figure 1(a). The three vertices of the small triangle in the center are pushed down slightly so that all dotted edges are reflex, but are not flippable. On the other hand, if self-intersection is permitted during flipping, it is possible to transform a polyhedron to the convex hull for the restricted case when all vertices are extreme vertices that appear on the boundary of the convex hull [Alboul 2003]. In a relevant development, Shewchuk [2005] proposes star splaying, a different kind of local transformation, which is capable of transforming any polyhedron, even with topological error, into the convex hull, but the splaying process is much more complex and less efficient than flipping.

Our work aims to better understand flipping for its simplicity and locality nature having a strong potential for efficient parallel computation. Towards this end, the paper has the following contributions:

- A novel flip algorithm, called Flip-Flop, to provably transform any star-shaped polyhedron to its convex hull.
- An algorithm, called ffHull, to construct the exact 3D convex hull using flipping that works well on both CPUs and GPUs.
- The first known algorithm, called ffRT, to construct the 2D regular triangulation on the GPU.

The Flip-Flop algorithm flips not only reflex edges as in the Lawson’s flip algorithm, but also convex ones (hence the “flop” part) when suitable. The salient point of Flip-Flop is its ability to dis-
cover automatically all the non-extreme vertices that do not appear on the boundary of the convex hull and to remove them, both using the local flipping operation. Though working for only star-shaped polyhedra, Flip-Flop is used as the key component for ffHull to construct the convex hull of a point set in $\mathbb{R}^3$ and for ffRT to construct the regular triangulation of a point set in $\mathbb{R}^2$, with provable correctness. Our algorithm allows nonrestrictive insertion of many vertices, both extreme and non-extreme, before any flipping of edges. Besides, by enforcing an invariant that the polyhedron is always star-shaped after each flip, which is achieved using only local checks, we ensure no self-intersection checks are needed. These not only maintain the simplicity and locality of the flip algorithm, but also map well to the massively parallel nature of the modern GPUs. Our extensive experiments show that ffHull and ffRT running on the CPU are as fast as the best known implementations on GPUs. Our extensive experiments show that ffHull and ffRT demonstrate their suitability for parallel computation by outperforming all known prior work, while remaining robust and exact.

The rest of the paper is organized as follows. Section 2 presents the Flip-Flop algorithm in $\mathbb{R}^3$. Section 3 introduces ffHull and Section 4 extends it to ffRT. Section 5 gives the experiment results of our algorithms on both the CPU and the GPU. Finally, Section 6 concludes the paper with some potential directions for future work. The proof of correctness of the Flip-Flop algorithm and thus of ffHull and ffRT is provided in Appendix A.

2 Flip-Flop Algorithm

We present the Flip-Flop algorithm that takes a star-shaped polyhedron in $\mathbb{R}^3$ as input and produces the convex hull of the points of the polyhedron.

2.1 Definitions

Let $S$ be a set of points in $\mathbb{R}^3$, assuming in general positions. The convex hull of $S$, denoted as $CH(S)$, is the collection of points where each is a convex combination of the points in $S$. This collection is also the smallest convex set containing $S$, and the boundary of $CH(S)$ is thus a convex polyhedron. Similarly, the convex hull for a set $R$ of rays, denoted as $CH(R)$, is the collection of points where each is a convex combination of some points on the rays of $R$. $CH(R)$ extends to infinity.

Given a point $s \in \mathbb{R}^3$ and three points $a, b, c \in S$, the cone of $ab$ w.r.t. $s$, denoted as $C_3(\triangle abc)$, is the convex hull of the three rays $\{sa, sb, sc\}$. Two cones overlap if they contain some common points not on their boundaries. Let $T$ be a polyhedron with vertices in $S$ and with faces that are triangles. We say $T$ is a star-shaped polyhedron w.r.t. $s$ if and only if $s$ is in the interior of $T$ and the cones of any two triangles of $T$ w.r.t. $s$ do not overlap each other. This means the boundary of a star-shaped polyhedron is entirely visible from $s$. $s$ is called the kernel of $T$.

In $T$, given an edge $e = \overrightarrow{ab}$ with endpoints $a$ and $b$, $e$ is incident to two triangles $t_1 = \triangle abc$ and $t_2 = \triangle bad$. $\{a, b\}$ is called the link of $e$. The two sides of a triangle are the two half-spaces defined by the plane containing the triangle. $e$ is a reflex edge w.r.t. $s$ if $c$ and $s$ lie on different sides of $t_2$; otherwise it is a convex edge. Indeed, since $T$ is star-shaped, $c$ and $s$ lie on two different sides of $t_2$ if and only if $d$ and $s$ lie on two different sides of $t_1$. Convexity and reflexivity are local properties of edges of $T$. On the other hand, if $T$ has no reflex edges, then it is the convex hull of its vertices.

Since $T$ is topologically a 2D triangulation, flips on $T$ can be classified into 3-1 flips, 2-2 flips and 1-3 flips [Edelsbrunner and Shah 1992]; see Figure 2. A 3-1 flip removes a vertex from $T$, while a 1-3 flip inserts a new vertex into $T$. For the discussion of Flip-Flop, we do not concern with 1-3 flips. Given an edge $e = \overrightarrow{ab} \in T$ with $\{a, b\}$ being its link, the induced subcomplex of $e$, denoted as $\sigma_e$, is the set of triangles in $T$ that span $\{a, b, c, d\}$. Clearly, flipping $e$ is a 2-2 flip if $|\sigma_e| = 2$. We call $e$ in this case a 2-2 edge. Similarly, $e$ is a 3-1 edge if $|\sigma_e| = 3$. We say that $e$ is flippable w.r.t. $s$ if firstly $s$ is outside the tetrahedron $abcd$, and secondly the union of the cones of the triangles of $\sigma_e$ is equal to $CH(\{s\cap a, sb, sc, sd\})$.

Otherwise, $e$ is unflippable. If $e$ is a 3-1 edge, the second condition is always true, while if $e$ is a 2-2 edge, the second condition is true if and only if any vertex among $\{a, b, c, d\}$ is outside the cone of the triangle formed by the other three vertices. If a 2-2 edge is flippable, we also call it 2-2 flippable for short. Similarly, a 3-1 edge that is flippable is called 3-1 flippable.

Flipping a flippable edge that is reflex turns it into a convex one, and vice versa. Moreover, the definition ensures that flipping a flippable edge of a star-shaped polyhedron w.r.t. $s$ results in a polyhedron that remains star-shaped w.r.t. $s$. In another view, no intersections are introduced during such flips, and the check for the flippability of an edge is purely local. As such, flipping can be used as a local operation to transform a star-shaped polyhedron into the convex hull of its vertices.

2.2 Flipping Criteria

Traditionally, we only flip a flippable edge $e$ if it is reflex. This flip increases the volume of $T$, and we refer to this flipping criterion as the V-criterion (V for Volume). Clearly, the convex hull has the maximum volume possible, and thus this flip algorithm is a hill climbing method. It works when all vertices in $T$ are extreme vertices, but possibly gets stuck at a local optimum when not all vertices are extreme vertices, as shown in Figure 1(b).

We can show that any 3-1 edge that is reflex is always flippable. On the other hand, any 2-2 edge that is reflex is flippable; otherwise it has a non-extreme vertex as its endpoint. If all these non-extreme vertices are removed, we would be able to flip $T$ to its convex hull. A direct way to remove a vertex is to delete all triangles incident to it and re-triangulate the resulting hole. This is difficult to do while still keeping $T$ star-shaped. Moreover, it is no longer a local operation when we try to avoid self-intersection. Instead, we observe that to remove a vertex, one can attempt to use 2-2 flips to first reduce its degree to 3; see Figure 3. Then, a 3-1 flip can be applied to complete the vertex removal. A relevant approach has been proposed by Ledoux et al. [2005], but the input needs to be a DT.

Based on the above mentioned observation, we introduce a new criterion for our flip algorithm. We flip a flippable edge as long as

![Figure 2: Flipping on a 3D polyhedron.](image-url)

![Figure 3: The star of $v$ as seen from the kernel point. To remove $v$, we successively flip $\overrightarrow{av}$ and $\overrightarrow{cv}$ to reduce the degree of $v$ to 3. Then, $v$ is removed by a 3-1 flip.](image-url)
it is incident to a non-extreme vertex. This decreases the degree of a non-extreme vertex in \( T \). However, this might lead to flipping an edge back and forth if multiple vertices are labeled as non-extreme. To avoid this, \( e \) is only flipped if that decreases the degree of the vertex with the smallest index among all non-extreme vertices in \( \{a, b, c, d\} \). We refer to this as the D-criterion (D for Degree).

### 2.3 Flip with V- and D-criterion

The Flip-Flop algorithm is designed based on both the V-criterion and the D-criterion. We follow the D-criterion to remove non-extreme vertices while at the same time we use the V-criterion to move closer to the convex hull. In order to apply the D-criterion, we analyze 2-2 unflippable edges to label non-extreme vertices; see Figure 4(a). Algorithm 1 shows the detailed pseudocode of Flip-Flop. The input of the algorithm is a star-shaped polyhedron \( T \) w.r.t. a given point \( s \). The output is \( CH(T) \), i.e. the convex hull of all vertices of \( T \). The algorithm repeatedly checks the edges in \( T \) and flips them if necessary. For efficiency, we use a queue \( Q \) to keep track of edges that need to be checked. An edge that has been checked need not be checked again unless its incident triangles are changed or one of its adjacent vertices is labeled as non-extreme. Each time we flip an edge, we push all the newly created edges into \( Q \). Note that an edge in \( Q \) is attached to its two incident triangles, and thus is considered outdated if any one of these triangles are removed from \( T \) due to some flips.

We now discuss Algorithm 1 in details. First of all, all the vertices of \( T \) are labeled as unknown (Line 1) to indicate that we do not know whether they are non-extreme vertices or not. All edges of \( T \) are pushed into \( Q \) for checking (Line 2). The loop in Line 3–20 repeats until no more edges in \( T \) need to be checked, by which \( T \) has no reflex edges and thus the algorithm reaches the required output. Let \( e = \overline{ab} \) be an edge popped in one iteration, and let \( \{c, d\} \) be its link. If \( e \) is outdated, we can just ignore it (Line 5).

First, consider the situation that \( e \) is a 3-1 edge (Line 7–10). Either \( a \) or \( b \) must lie inside the cone of the triangle formed by the other 3 vertices; see Figure 4(b). Without loss of generality, let \( a \) be inside \( C_s(\triangle bcd) \). If \( a \) is labeled as non-extreme or \( e \) is reflex (which also implies that \( a \) is non-extreme), we flip \( e \) to remove \( a \). In both cases, \( e \) is flippable (see Appendix A).

Second, consider the situation that \( e \) is a 2-2 edge (Line 11–20). There are two cases, depending on whether the union of the cones of the triangles of \( \sigma_e \) is equal to \( CH(\{\overline{a\bar{a}}, \overline{a\bar{b}}, \overline{a\bar{s}}, \overline{a\bar{d}}\}) \) or not (Line 12). If this condition is true, then \( e \) fulfills the second condition of being flippable. We flip \( e \) either if \( e \) is reflex and no vertices among \( \{a, b, c, d\} \) are labeled as non-extreme vertices (Line 13–14), or if \( e \) is a 2-2 edge that is reflex and unflippable. In both cases, \( e \) is a non-extreme vertex since it lies inside the tetrahedron \( s\overline{abcd} \).

Algorithm 1 Flip-Flop

**Input:** a star-shaped polyhedron \( T \) w.r.t. \( s \)

**Output:** \( CH(T) \)

1: label all vertices of \( T \) as unknown
2: \( Q \leftarrow \{ e \mid e \text{ is an edge of } T \} \)
3: while \( Q \) is not empty do
4: \( e \leftarrow Q\text{.pop()} \)
5: if \( e \) is outdated then continue
6: let \( e = \overline{ab}, \{c, d\} \) be its link and \( x \in \{a, b, c, d\} \) be the non-extreme vertex with smallest index
7: if \( e \) is a 3-1 edge then
8: let \( a \) be inside \( C_s(\triangle bcd) \)
9: if \( a \) is labeled as non-extreme or \( e \) is reflex then
10: flip \( e \); \( Q \leftarrow \{bc, cd, db\} \)
11: else if \( e \) is a 2-2 edge
12: if \( a \notin C_s(\triangle bcd) \) and \( b \notin C_s(\triangle acd) \) then
13: if \( x \) does not exist and \( e \) is reflex then
14: flip \( e \); \( Q \leftarrow \{ac, bc, ad, bd\} \)
15: else if \( x \in \{a, b\} \) and \( x \notin CH(\{a, b, c, d\}) \) then
16: flip \( e \); \( Q \leftarrow \{ac, bc, ad, bd\} \)
17: else if \( e \) is reflex then
18: let \( a \) be inside \( C_s(\triangle bcd) \)
19: label \( a \) as non-extreme
20: \( Q \leftarrow \{e' \mid e' \text{ is an edge of } T \text{ with } a \text{ as an endpoint} \} \)

![Figure 4: (a) \( \overline{ab} \) is a 2-2 edge that is reflex and unflippable. (b) \( \overline{ab} \) is a 3-1 edge that is reflex. In both cases, \( a \) is a non-extreme vertex since it lies inside the tetrahedron \( s\overline{abcd} \).](image-url)
not be extreme vertices, or use the traditional flipping algorithm which cannot always produce the convex hull. More specifically, in the correctness proof for CudaHull, the authors claim that the algorithm flips all concave edges, while in fact their algorithm does not flip edges that lead to self-intersections. As a result, in some cases the output polyhedron is not a convex hull because some concave edges remain. Recently, Gao et al. [2012] propose the gHull algorithm, using the digital Voronoi diagram to construct an approximation followed by several other fixing steps including the use of star splaying [Shewchuk 2005] on the GPU. Due to the use of an approximation from the digital space, their algorithm cannot handle point sets from a non-uniform distribution very efficiently.

The Flip-Flop algorithm is a powerful tool to transform any star-shaped polyhedron into its convex hull. We introduce a new algorithm, termed ffHull, that uses Flip-Flop to construct the convex hull of a set of points in $\mathbb{R}^3$. The main idea is to quickly construct a star-shaped polyhedron $T$ that is close to $CH(S)$ by an incremental insertion approach, followed by applying Flip-Flop.

### 3.1 Algorithm

**Algorithm 2 ffHull**

**Input:** a set $S$ of points in $\mathbb{R}^3$

**Output:** $CH(S)$

1. $t \leftarrow \{a, b, c, d\}$
2. $v \leftarrow \text{centroid of } T$
3. $\Delta \in S$ such that $v \notin \Delta$
4. $\Delta \notin S$ and $\Delta \notin \text{boundary of } T$
5. $\Delta$ is a triangle in $S$
6. $\Delta$ is associated with $T$
7. $\Delta$ is the centroid of $T$
8. $\Delta$ is the convex hull of $T$
9. $\Delta$ is a point in $S$
10. $\Delta$ is the furthest point associated to $T$
11. $\Delta$ is the furthest point associated to $T$
12. $\Delta$ is the centroid of $T$

Algorithm 2 shows the pseudocode of ffHull. There are two stages: constructing a star-shaped polyhedron (Line 1–11), and transforming it to the convex hull using Flip-Flop (Line 12). In the first stage, from the initial star-shaped polyhedron $T$ with 4 extreme vertices (Line 1), we incrementally process input points: an input point is either inserted to become a vertex of the polyhedron or removed if found to be inside the polyhedron constructed so far. Specifically, during the process, we use the centroid $s$ of the initial star-shaped polyhedron as the kernel point (Line 2). For each point $p$ in $S$, we associate $p$ with a unique $\Delta$ of $T$ if $p$ is inside the cone of $\Delta$. We remove $p$ if it is inside the tetrahedron $sabc$, i.e., when $p$ and $s$ lie on the same side of $\Delta$, since it is a non-extreme point. In subsequent loop, $T$ is grown by inserting the furthest point $v$ associated to each triangle $t$ in $T$ into $T$ (Line 5–7). This is done by replacing $t$ with three new triangles. Each point inserted into $T$ is removed from $S$. Each insertion splits a cone $C_i(t)$ into three new disjoint cones. This guarantees that $T$ is still star-shaped after the insertion. Line 8–11 update the triangle each point $p \in S$ is associated to, or remove $p$ if it is a non-extreme point. This “growing” process is repeated until $S$ is empty. In the second stage, we simply apply the Flip-Flop algorithm to transform $T$ to $CH(S)$.

We have two notes for the above algorithm. Firstly, Flip-Flop actually works for any star-shaped polyhedron. It is thus not necessary to always find the furthest point to insert into $T$, even though doing so helps to remove non-extreme points in $S$ quickly. In fact, it is costly to find such furthest points because of numerical error. In practice, we often choose the almost furthest points to insert to construct the star-shaped polyhedron. Secondly, the algorithm presented above is just one approach to use Flip-Flop to construct the convex hull. Another possibility is to alternate between inserting points and flipping in multiple iterations. That is, however, observed to have lower parallelism during flipping in our experiments.

### 3.2 GPU Implementation

We use CUDA for our implementation of ffHull on the GPU. The polyhedron is represented as an array of triangles, each containing the indices of its three vertices and the indices of the triangles sharing its three edges. This is similar to the data structure of triangulation used in CGAL [Boissonnat et al. 2002] and GPU-DT [Qi et al. 2012]. Furthermore, some auxiliary arrays are also used for intermediate computation. For example, we need an array to store the index of the furthest point for each triangle and an array to store for each point the index of the triangle it is associated to. The arrays are dynamically expanded rather than pre-allocated since usually only a small number of points appear in the polyhedron.

We use two techniques to simplify (and at the same time optimize) the implementation of ffHull. Let $\text{orient}(p, t)$ be the determinant used to determine whether the point $p$ lies beneath or above the triangle $t$. The first technique is to maintain the orientation of each triangle $t$ in the polyhedron so that the kernel point $s$ is beneath $t$. With this, to perform Line 11 of Algorithm 2, we only need to compute $\text{orient}(p, t)$. Similarly, we can check the reflexivity of an edge, without using $s$. The second technique is to reuse $\text{orient}(p, t)$, which is the volume of the tetrahedron formed by $p$ and the triangle $t$, when finding the furthest point to $t$ instead of actually computing the distance.

The details on the implementation of ffHull is as follows. In the first stage, constructing a star-shaped polyhedron, we have four major CUDA kernels. The first kernel (Line 3 of Algorithm 2), with one thread processing one input point, finds the triangle in the initial triangulation $T$ that is associated to and at the same time participates in the search for the furthest point of each triangle. We use two arrays in the global memory to store for each triangle the furthest point and its distance, with the initial value being 0. The thread processing and associating $p$ to a triangle $t$ uses $\text{orient}(p, t)$ to judge whether $p$ is beneath $t$ or not, and marks $p$ as deleted if it is beneath. Otherwise, the distance from $p$ to $t$ (actually $\text{orient}(p, t)$ is used) is compared with the currently recorded value and if the new distance is larger, $p$ is recorded as the furthest point of $t$, and the distance is updated.

The second kernel (Line 5–7), with one thread processing one triangle $t$, inserts one point associated to $t$ into the polyhedron. A point being inserted into $t$ replaces it with three new triangles. The first one is stored in the original slot, while the other two are appended into the end of the array of triangles. Then, the third kernel, with one thread processing one triangle, updates the full adjacency information of all new triangles and those adjacent to them. A separate kernel is necessary here since updating directly in the second kernel may create memory read and write conflict. The fourth kernel (Line 8–11), with one thread processing one point, updates $p$’s associated triangle if $p$ is still outside $T$. For $p$ with $t$ being the previous associated triangle, we first read the slot in the triangle array that previously stored $t$, which now stores the first new triangle. From its adjacency information, we obtain the other two new triangles. Among these three, we identify the triangle $t$ that is now associated to. Then the furthest point for $t$ is updated similar to the first kernel mentioned above. Note that in the first and fourth kernels, since the threads are executed in parallel, we do not always get the furthest point for each triangle. However, this approach is efficient.
without compromising on the correctness. During this stage, the index of the associated triangle of a point is set to \(-1\) if the point is found to be inside the polyhedron, and that point will not be revisited in the subsequent computation.

In the second stage, the Flip-Flop stage, the flipping is done in multiple iterations. In each iteration, we use two kernels, a checking kernel and a flipping kernel, to perform multiple flips in parallel, similar to the technique described in [Qi et al. 2012] and [Navarro et al. 2011]. In the checking kernel, we assign one thread to one triangle to check if one of its edges can be flipped (based on the V- criterion). The difficulty is that the induced sub-complexes of some edges share some triangles, thus the flips are conflicting and cannot be done in the same iteration. To avoid this, if a thread in charge of \(\triangle abc\) wants to flip the edge \(e = \overline{ab}\), it uses the atomic minimum operation to label the triangles of \(\sigma_e\) with the index of \(\triangle abc\). In the flipping kernel, we also assign one thread to one triangle. The thread in charge of \(\triangle abc\) only flips \(e\) if the triangles of \(\sigma_e\) are still labeled with the index of \(\triangle abc\). This guarantees no conflicting flips are performed in the same iteration, and in each iteration at least one flip can be done. Note that only up to three threads write to the same memory location during the labelling, so the use of the atomic operation does not affect the efficiency much.

### 3.3 Exact Computation and Robustness

The only predicate we use, the 3D orientation predicate, is adapted from the exact predicate of Shewchuk [1997]. Instead of using a multiple-stages adaptive arithmetic, we only use two stages: a fast computation with all arithmetic operations being done using native floating-point numbers, and an exact computation with all arithmetic operations being fully expanded into arrays of floating-point numbers. To verify whether the fast computation gives the correct sign of the determinant being computed or not, we use the forward error analysis approach described by Shewchuk. The error bounds are pre-computed using a single CUDA kernel, and stored for later used by all kernels that need exact computation.

The exact computation code requires a lot of registers and local memory, so each kernel that needs exact computation is split into two kernels. The first one performs only fast computations, and uses the error bounds to determine whether it requires exact computation or not. In second kernel, only the threads that need exact computation are active. By doing this, the first kernel requires less registers and local memory, and thus can run with higher parallelism. The second kernel, on the other hand, has very little work to do.

We also use the SoS technique of Edelsbrunner and Mücke [1990] to handle degeneracy. The implementation is as described in their original paper. All the computation of the subdeterminants are done with exact arithmetic. One small note is that although SoS requires computing many different subdeterminants, the CUDA code is written in such a way that the call to exact predicate appears only once, inside a loop with different subdeterminants being passed to it. Consequently, we can force inline the exact predicate function without worrying about the kernel code being too large to compile.

With the use of exact predicates and SoS, we can guarantee the exactness and robustness of our implementation. The only source of inexact computation is when computing the kernel point \(s\). If the initial tetrahedron is almost flat, its centroid computed inexacty can lie outside. By carefully choosing the first four extreme points, we make sure they are far away from each other to avoid the undesirable situation, unless all the input points are almost co-planar.

### 4 Extension to Regular Triangulation

The regular triangulation of a set \(S\) of weighted points in \(\mathbb{R}^3\) is often computed using the incremental insertion approach by Edelsbrunner and Shah [1992]. On the other hand, the regular triangulation is closely related to the convex hull of \(S\) when lifted into \(\mathbb{R}^3\). The lift map is \(\pi: (x, y) \mapsto (x, y, x^2 + y^2 - w)\) where \(w\) is the weight of the point. Let the lift point set be \(S'\). The lower hull of the convex hull of \(S'\) is equivalent to the regular triangulation of \(S\). Existing GPU algorithms to construct convex hull are not suitable for constructing the regular triangulation for several reasons. First, most of the points are expected to appear in the final triangulation, so the filtering method presented by Tang et al. [2012] is not very useful. Second, the range of the \(z\)-coordinates is very large compared to that of the \(x\)- and \(y\)-coordinates, thus using the digital Voronoi diagram as proposed by Gao et al. [2012] is not very suitable.

The Flip-Flop algorithm does not have any of the disadvantages mentioned above. Thus we extend ffHull to ffRT, a new algorithm to compute the regular triangulation of a set of weighted points in \(\mathbb{R}^3\). Conceptually, we lift the input point set \(S\) into \(\mathbb{R}^3\) to get \(S'\), simulate ffHull on \(S'\) to get \(\mathcal{CH}(S')\) and finally return its lower hull as output. The actual implementation of ffRT is adapted from that of ffHull with some modification to improve the efficiency. We initialize the star-shaped polyhedron \(T\) with a tetrahedron containing 3 extreme vertices \(\{u, v, w\}\) of \(S'\) and a virtual vertex \(\hat{p}\) with the \(z\)-coordinate being \(+\infty\), and thus the kernel point \(\hat{s}\) is the centroid of \(\{u, v, w\}\) but with the \(z\)-coordinate being \(+\infty\). This makes all upper hull triangles being incident to \(\hat{p}\) and thus can easily be removed before we output the triangulation.

During the growing process, there are two kinds of triangles in \(T\): virtual triangles that are incident to \(\hat{p}\) and real triangles that are not. The cone of a real triangle \(\triangle abc\) is the space bounded by the three vertical planes going through the edges \(\overline{ab}, \overline{bc},\) and \(\overline{ca}\). The cone of a virtual triangle \(\triangle ab\hat{p}\) can be seen as the space bounded by the vertical planes going through \(ab\) and \(\hat{p}\) intersecting with the half space defined by the vertical plane going through \(ab\) that does not contain \(\hat{s}\). Similarly, in the flipping stage there are three kinds of edges: internal edges that are incident to two real triangles; virtual edges to two virtual triangles; and silhouette edges to one real triangle and one virtual triangle. A virtual edge is always flippable, and checking its reflexivity is done using a 2D orientation check. A silhouette edge is always convex, while an internal edge is treated as a normal edge in ffHull.

### 5 Experimental Results

Our algorithms are implemented on the GPU using the CUDA programming model by Nvidia, and can easily be ported to OpenCL. All the experiments are conducted on a PC with an Intel i7 2600K 3.4GHz CPU, 16GB of DDR3 RAM and an Nvidia GTX 580 Fermi graphics card with 3GB of video memory. All implementations are compiled with all optimizations enabled.

#### 5.1 Convex Hull

We compare the performance of our implementations of the ffHull algorithm on both the CPU and the GPU with the two fastest convex hull implementations on the CPU, qHull and CGAL, as well as with gHull, the recently published GPU algorithm by Gao et al. [2012]. Both CGAL and gHull use exact predicates, similar to our implementations, while qHull produces a convex hull with “thick” facets. According to our experiment results, qHull runs faster than CGAL in most cases, so all the speedup reported in this subsection is with respect to qHull.
In Figure 6(b), we present the total number of points on the convex hull. First, we show an experiment on some representative point distributions. We generate points randomly with coordinates between [0.0, 1.0]. Following Gao et al.'s work, we use four representative distributions of points to test: a cube, a ball of radius of 0.5, a sphere of thickness of 0.01 and a box with thickness of 0.01. The cube and the box distributions have very few points on the convex hull, while the ball and the sphere distributions have many. The box and the sphere distributions have most points close to the convex hull, making it difficult for the Quickhull algorithm to remove non-extreme points.

Figure 5 shows the running time of the CPU algorithms and the speedup of the GPU algorithms when compared to qHull. Clearly, our CPU implementation of ffHull outperforms CGAL in all the tested distributions. Our CPU version of ffHull is also as fast as qHull, while always producing an exact convex hull rather than just an approximation like qHull does. On the other hand, our GPU implementation of ffHull achieves up to 30x speedup compared to qHull, and is faster than gHull in all the cases. As mentioned by Gao et al. [2012], their algorithm does not perform well in the cube and the box distributions due to the use of approximation in the digital space. Our ffHull is not affected by this, and thus runs much faster than gHull for these cases.

We also use models of over a million points from the Stanford 3D scanning repository to test the ffHull algorithm; see Table 1. These models have very few points on the convex hull, while most other points are distributed near the surface, with many points being coplanar. Once again, our ffHull implementation on the GPU outperforms gHull in all cases. The CPU implementation, though slower than qHull due to a large amount of exact computation required, still outperforms CGAL.

<table>
<thead>
<tr>
<th>Model</th>
<th># Points (millions)</th>
<th>qHull</th>
<th>CGAL</th>
<th>ffHull-CPU</th>
<th>ffHull-GPU</th>
<th>gHull</th>
<th>gHull-GPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asian dragon</td>
<td>3.6</td>
<td>540</td>
<td>1181</td>
<td>997</td>
<td>214</td>
<td>141</td>
<td></td>
</tr>
<tr>
<td>Thai statue</td>
<td>5.0</td>
<td>692</td>
<td>1538</td>
<td>1240</td>
<td>180</td>
<td>125</td>
<td></td>
</tr>
<tr>
<td>Lucy</td>
<td>13.3</td>
<td>1884</td>
<td>4488</td>
<td>3604</td>
<td>297</td>
<td>263</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Running time on different 3D models.
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A Proof of Correctness

We show that the Flip-Flop algorithm can successfully transform any star-shaped polyhedron in \( \mathbb{R}^3 \) into its convex hull. It then follows that the fHull algorithm computes correctly the convex hull of a point set. In the following discussion, let \( T \) be a star-shaped polyhedron in \( \mathbb{R}^3 \) with respect to a given kernel point \( s \). Let \( F(v) \) denote the set of triangles incident to the vertex \( v \) of \( T \), and \( L(v) \) be the set of vertices and edges in \( F(v) \) excluding those incident to \( v \). \( F(v) \) and \( L(v) \) are called the fan and the link of \( v \) respectively.

Given an edge \( e = \overrightarrow{ab} \in T \) with \( [c, d] \) being its link, the induced subcomplex of \( e \), denoted as \( \sigma_e \), is the set of triangles in \( T \) that span \([a, b, c, d]\). We use \( C_s(F(v)) \) to denote \( \bigcup_{t \in F(v)} C_s(t) \), and similarly for \( C_s(\sigma_e) \).

A.1 Properties of Star-Shaped Polyhedra

**Lemma 1.** For any vertex \( v \) of \( T \), none of the vertices of \( T \setminus L(v) \) except \( v \) lies inside \( C_s(F(v)) \). Moreover, \( v \) does not lie on the boundary of \( C_s(F(v)) \).

**Proof.** This follows immediately from the definition of the star-shaped polyhedron.

**Lemma 2.** Any 3-1 edge \( e = \overrightarrow{ab} \) of \( T \) that is reflex is flippable.

**Proof.** Let the link of \( e \) be \( \{c, d\} \). Without loss of generality, let \( \sigma_e = \{abc, \triangle bad, \triangle acd\} \); see Figure 4(b). Since \( e \) is reflex, \( s \) and \( d \) lie on different sides of \( \triangle abd \), so \( s \) is outside the tetrahedron \( abcd \). On the other hand, by Lemma 1, \( a \) lies inside \( C_s(\sigma_e) \), and thus inside \( CH(\{\overrightarrow{sa}, \overrightarrow{sb}, \overrightarrow{sc}, \overrightarrow{sd}\}) \). Therefore, \( e \) is flippable.

**Lemma 3.** Any 2-2 unflippable edge \( e = \overrightarrow{ab} \) is reflex that is incident to a non-extreme vertex.

**Proof.** Let the link of \( e \) be \( \{c, d\} \). Since \( e \) is reflex, \( s \) and \( d \) lie on different sides of \( \triangle abd \), therefore \( s \) is outside the tetrahedron \( abcd \). As such, by the definition of the unflippable edge, the union of \( C_s(\triangle abc) \) and \( C_s(\triangle bad) \) must not be equal to \( CH(\{\overrightarrow{sa}, \overrightarrow{sb}, \overrightarrow{sc}, \overrightarrow{sd}\}) \); see Figure 4(a). This implies that either \( a \) is inside \( C_s(\triangle bcd) \) or \( b \) is inside \( C_s(\triangle acd) \), which results in a non-extreme vertex since \( e \) is reflex.

A.2 Star-Shaped Polyhedra with all Extreme Vertices

We prove that any star-shaped polyhedron with all vertices being extreme vertices can be transformed into its convex hull by flippings with the V-criterion. Using the invariant that \( T \) is still star-shaped w.r.t. \( s \) after flipping any flippable edge, we show that in this case, any reflex edge is in fact flippable, which implies that we can flip all the reflex edges of \( T \) to get to its convex hull.

A.3 Star-Shaped Polyhedra with Non-Extreme Vertices

We introduce several new concepts for the proofs in this section. Let \( v, a, b, c \) be different vertices of \( T \). The cone \( C_s(\triangle abc) \) is called the cover of \( v \) if \( v \) lies inside it. It is then called the minimal cover of \( v \) if no other vertices of \( T \) lie inside it. Note that \( v \) might have more than one minimal cover. We say \( v \) is locally covered if it has a cover formed by three vertices on its link; see Figure 8. We show that any non-extreme vertex \( v \) is locally covered, and we can always find a flippable edge incident to it. Thus we can use the D-criterion to decrease the degree of \( v \) and eventually remove it.

**Lemma 5.** If \( v \) is a non-extreme vertex, then \( v \) is locally covered.

**Proof.** By contradiction, assume that \( v \) is not locally covered. This means no cone of three vertices of \( L(v) \) is a cover of \( v \). We construct \( CH(L(v)) \) and the line \( vs \), as shown in Figure 9(a). By Lemma 1, \( v \) is inside \( C_s(F(v)) \), so the line \( vs \) must intersect \( CH(L(v)) \) at two triangles \( t_1 \) and \( t_2 \). Note that \( t_1 \) and \( t_2 \) are possibly the same triangle when \( L(v) \) has only 3 vertices. Since \( C_s(t_1) \) and \( C_s(t_2) \) are not covers of \( v, v \) and \( s \) must lie on the same side of both triangles with \( v \) being further. Without loss of generality, let the intersection of \( t_1 \) and the line \( vs \) be nearer to \( s \) than that of \( t_2 \), and let \( H \) be the plane through \( s \) and parallel to \( t_2 \). We prove that all vertices of \( T \) lie on the half-space of \( H \) not containing \( v \). Let \( R_t \) be this half-space, clearly \( CH(L(v)) \) lies inside \( R_t \). Consider any vertex \( p \) in \( T \) other than \( v \) and those in \( L(v) \). Consider the plane through \( p, v \) and \( s \), the half-plane defined by \( vs \) and containing \( p \) must intersect \( L(v) \) at a point \( q \). The point \( p \) must lie outside the angle \( vsq \); otherwise it falls into \( C_s(F(v)) \).
Theorem 2. The Flip-Flop algorithm computes the convex hull of any star-shaped polyhedron.

Proof. The Flip-Flop algorithm definitely terminates, since each flip performed either increases the volume of \( T \) or decreases the degree of a non-extreme vertex with increasing the degree of any other non-extreme vertex with smaller index. The algorithm flips all flippable edges, and if there is any unflippable edge, by Lemma 3, we can identify a non-extreme vertex, and by Lemma 7 and Lemma 8, we can flip and remove it. As such, when the algorithm terminates, the result is the convex hull.

Theorem 2 promises the correctness of Flip-Flop in sequential execution. When we execute it in parallel, conflicting flippings may break the validity of the star-shaped polyhedron. By using atomic operations, we avoid this problem. Since the Flip-Flop algorithm follows the V- and D- criterion and the number of non-extreme vertices is limited, the algorithm always terminates after removing all non-extreme vertices and flipping all reflex edges. Thus, the proof of Theorem 2 applies to the parallel execution as well.

Theorem 3. The fFlull algorithm computes the convex hull of any point set in \( \mathbb{R}^3 \).

Proof. The point insertion process in the first phase of fFlull generates a star-shaped polyhedron from the input points, with those certainly non-extreme vertices being excluded. The rest of the proof follows from Theorem 2.