UIT2206: The Importance of Being Formal

Assignment Week 08:
Midterm Review

Traditional Logic

1. Consider the following syllogism:

   All humans are mortal.
   All Greeks are humans.
   Therefore, all Greeks are mortal.

   Identify all propositions and all categorical terms in this syllogism. Define a suitable universe to give meaning to all terms.

2. Consider the following syllogism:

   All slack track systems are caterpillar systems.
   All Christie suspension systems are slack track systems.
   Therefore, **some** Christie suspension systems are caterpillar systems.

   Give a model in which this syllogism does not hold.

3. Consider the following statements:

   The only articles of food, that my doctor allows me, are such as are not very rich.
   Nothing that agrees with me is unsuitable for supper.
   Wedding cake is always very rich.
   My doctor allows me all articles of food that are suitable for supper.
   Therefore, wedding cakes do not agree with me.

   Identify all categorical terms in these statements. Define a suitable universe to give meaning to all terms.

4. Is the proposition

   `Some humans are humans`

   valid? Use the semantics of categorical propositions in your argument.

Answer

`Some humans are humans` is not valid, because we can choose a model $\mathcal{M}$ such that $\texttt{humans}^{\mathcal{M}} = \emptyset$. In this case, $\texttt{humans}^{\mathcal{M}} \cap \texttt{humans}^{\mathcal{M}} = \emptyset$ and $(\texttt{Some humans are humans})^{\mathcal{M}} = F$, which contradicts validity.

Propositional Logic

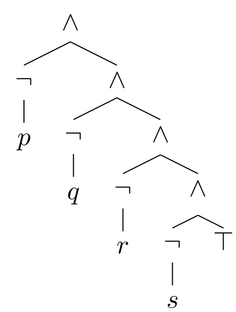
5. Draw the parse tree for the following formula:

$$((\neg p) \wedge ((\neg q) \wedge ((\neg r) \wedge ((\neg s) \wedge \top))))$$

List all sub-formulas of the expression.

Solution

The parse tree looks as follows:



There are 13 subformulas:

- $((\neg p) \wedge ((\neg q) \wedge ((\neg r) \wedge ((\neg s) \wedge \top))))$
- $(\neg p)$
- p
- $((\neg q) \wedge ((\neg r) \wedge ((\neg s) \wedge \top)))$
- $(\neg q)$
- q
- $((\neg r) \wedge ((\neg s) \wedge \top))$
- $(\neg r)$
- r
- $((\neg s) \wedge \top)$

- $(\neg s)$
- s
- $\top$

6. According to the operator precedences in Convention 1 (page 7), the following formula has a unique reading.

$$\neg p \wedge q \to \neg r \vee \neg p \to r$$

Indicate this reading by writing all parentheses, according to Definition 1.

Solution

$$(((\neg p) \wedge q) \to (((\neg r) \vee (\neg p)) \to r))$$

7. Consider the special case of propositional logic where the set of atoms A is empty, and where there are no binary operations. Is the set of resulting formulas still infinite? Describe the set of resulting formulas using English or a formal notation such that the reader understands what formulas it contains.

Solution

The formulas in this special case are either $\top$, or $\bot$, or arbitrarily long sequences of $\neg$ followed by $\top$ or $\bot$. One could write this set (by appealing to the readers intelligence to spot the pattern) as follows:

$$\{\top, \bot, \neg\top, \neg\bot, \neg\neg\top, \neg\neg\bot, \ldots\}$$

8. Give the result of evaluating the following formula with respect to the valuation $v : v(p) = F, v(q) = T, v(r) = T, v(s) = F$.

- $(\neg q \vee r) \wedge (r \to \neg q)$
- $p \to q \to s$
- $(\bot \to \bot) \wedge (\top \to \top)$

Solution

F, T, T

9. Is the following formula valid?

$$(p \wedge \neg q) \to (q \vee \neg p)$$

Show how you arrived at your answer.

Solution

Validity would require that the last column of the following truth table only contains T:

p	q	$p \wedge \neg q$	$q \vee \neg p$	$(p \wedge \neg q) \to (q \vee \neg p)$
F	F	F	T	T
F	T	F	T	T
T	F	T	F	F
T	T	F	T	T

Note that the last column does not only contain T; there is one F. Therefore, the formula is not valid.

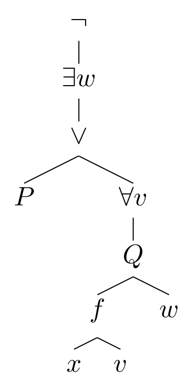
Predicate Logic

10. Let P be a predicate symbol of arity 0, Q be a predicate symbol of arity 2, f be a function symbol of arity 2, and g a function symbol of arity 1. Consider the formula

$$\phi = \neg(\exists w((P \vee (\forall v Q(f(x, v), w)))))$$

(a) Draw the parse tree of ϕ.

Solution



(b) Indicate the free and bound variables in this parse tree.

Solution

The variable occurrences of ϕ from left to right:

- x: free
- v: bound by $\forall v$
- w: bound by $\exists w$

(c) Compute $[x \Rightarrow g(f(x, x))]\phi$.

Solution

$$[x \Rightarrow g(f(x,x))]\phi =$$
$$\neg(\exists w((P \vee \forall v Q(f(g(f(x,x)),v),w))))$$

(d) Let ϕ be the sentence

$$\forall x \forall y \exists z (R(x,y) \to R(y,z))$$

where R is a predicate symbol of arity 2.

i. Let $A = \{a,b,c,d\}$ and $R^{\mathcal{M}} = \{(b,c),(b,b),(b,a)\}$. Does $\mathcal{M} \models \phi$ hold?
Justify your answer.

Solution

Due to the semantics of $\forall$ and $\exists$, for $\mathcal{M} \models \phi$ to hold we need to find for all combinations of values x, y in A for which $R(x,y)$ holds, a value z such that $R(y,z)$ holds.
For the combination $x = b$ and $y = c$, there is no such a value z, and thus, $\mathcal{M} \not\models \phi$.

ii. Let $A' = \{a,b,c\}$ and $R^{\mathcal{M}'} = \{(b,c),(a,b),(c,b)\}$. Does $\mathcal{M}' \models \phi$ hold? Justify your answer.

Solution

For $\mathcal{M}' \models \phi$ to hold we need to find for all combinations of values x, y in A' for which $R(x,y)$ holds, a value z such that $R(y,z)$ holds. For $(x,y) = (b,c)$, we find $z = b$, for $(x,y) = (a,b)$, we find $z = c$, and for $(x,y) = (c,b)$, we find $z = c$. For any other values of x, y, $R(x,y)$ does not hold, and thus $R(x,y) \to R(y,z)$ holds regardless whether $R(y,z)$ holds.
Thus, $\mathcal{M}' \models \phi$.

(e) Show that for any two sentences ϕ and ψ in predicate calculus, we have $\models \phi \to \psi$ whenever $\phi \models \psi$.

Solution

To show, according to the meaning of "whenever" in English:

$$\phi \models \psi \text{ implies } \models \phi \to \psi$$

Assume $\phi \models \psi$. This means according to the definition of entailment (Definition 2.20), that any model $\mathcal{M}$ that satisfies ϕ also satisfies ψ (*).

To show: $\models \phi \to \psi$, which means according to the definition of entailment that $\phi \to \psi$ holds in all models. Thus, according to the definition of satisfaction, we need to show that for any model $\mathcal{M}$, if $\mathcal{M} \models \phi$ holds then $\mathcal{M} \models \psi$ holds (**).

Let $\mathcal{M}$ be an arbitrary model. Case 1: $\mathcal{M}$ does not satisfy ϕ. In this case (**) trivially holds. Case 2: $\mathcal{M}$ satisifies ϕ. From (*), we can conclude $\mathcal{M} \models \psi$, and thus (**) holds.

(f) (repeated below as (g)) Is the sentence $\forall x P(x) \vee \forall x Q(x)$ entailed by $\forall x(P(x) \vee Q(x))$? Justify your answer.

(g) Is the sentence $\forall x P(x) \vee \forall x Q(x)$ entailed by $\forall x(P(x) \vee Q(x))$? Justify your answer.

Solution

No. According to the definition of entailment, for $\forall x(P(x) \vee Q(x)) \models \forall x P(x) \vee \forall x Q(x)$ to hold, any model that satisifies $\forall x(P(x) \vee Q(x))$ must also satisfy $\forall x P(x) \vee \forall x Q(x)$ (*).

Let $\mathcal{M}$ be a model with a universe of two elements, a and b. Let $P = \{a\}$ and $Q = \{b\}$. This model clearly satisfies $\forall x(P(x) \vee Q(x))$, but it does not satisfy $\forall x P(x) \vee \forall x Q(x)$. Thus there is a model $\mathcal{M}$ that satisfies $\forall x(P(x) \vee Q(x))$, but not $\forall x P(x) \vee \forall x Q(x)$, which contradicts (*). Thus (*) does not hold.

(h) Consider the sentences

$$
\begin{aligned}
\phi_1 &::= \forall x P(x, x) \\
\phi_2 &::= \forall x \forall y (P(x, y) \to P(y, x))
\end{aligned}
$$

Show $\phi_1 \not\models \phi_2$.

Solution

It suffices to find a model in which ϕ_1 holds, but not ϕ_2. Consider $\mathcal{M}$ with a universe $A = \{a, b\}$ and $P^{\mathcal{M}} = \{(a, a), (b, b), (a, b)\}$. In this model, ϕ_1 holds. However, ϕ_2 does not hold; $(a, b) \in P^{\mathcal{M}}$, but $(b, a) \notin P^{\mathcal{M}}$.

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