03—Propositional Logic I

The Importance of Being Formal

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Beyond Traditional Logic

Not just sets

How to express this using traditional logic?

- "$1 + 1 = 3$"
- "The sun is shining today."
- "Earth has more mass than Mars."

Arguments as Propositions

How to formalize a proposition of the form

If p_1 then p_2?

Atoms

Anything goes

We allow any kind of proposition, for example “The sun is shining today”.

Convention

We usually use p, q, p_1, etc, instead of sentences like “The sun is shining today”.

Atoms

More formally, we fix a set A of propositional atoms.

Meaning of Atoms

Models assign truth values

A *model* assigns truth values (F or T) to each atom.

More formally

A model for a propositional logic for the set A of atoms is a mapping from A to $\{T, F\}$.

How do you call them?

Models for propositional logic are called *valuations*.

Examples

Example

Some valuation Let $A = \{p, q, r\}$. Then a valuation v_1 might assign p to T, q to F and r to T.

More formally

$p^{v_1} = T, q^{v_1} = F, r^{v_1} = T$.
write $v_1(p)$ instead of p^{v_1}

Building Propositions

We would like to build larger propositions, such as arguments, out of smaller ones, such as propositional atoms. We do this using *operators* that can be applied to propositions, and yield propositions.

Unary Operators

Let p be an atom.

All possibilities

The following options exist:

1. $p^v = F$: $(op(p))^v = F$. $p^v = T$: $(op(p))^v = F$.
2. $p^v = F$: $(op(p))^v = T$. $p^v = T$: $(op(p))^v = T$.
3. $p^v = F$: $(op(p))^v = F$. $p^v = T$: $(op(p))^v = T$.
4. $p^v = F$: $(op(p))^v = T$. $p^v = T$: $(op(p))^v = F$.

The fourth operator *negates* its argument, T becomes F and F becomes T. We call this operator *negation*, and write $\neg p$ (pronounced “not p”).

Nullary Operators are Constants

The constant $\top$

The constant $\top$ always evaluates to T, regardless of the valuation.

The constant $\bot$

The constant $\bot$ always evaluates to F, regardless of the valuation.

Binary Operators: 16 choices

p	q	$op_1(p,q)$	$op_2(p,q)$	$op_3(p,q)$	$op_4(p,q)$
F	F	F	F	F	F
F	T	F	F	F	F
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_5(p,q)$	$op_6(p,q)$	$op_7(p,q)$	$op_8(p,q)$
F	F	F	F	F	F
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_9(p, q)$	$op_{10}(p, q)$	$op_{11}(p, q)$	$op_{12}(p, q)$
F	F	T	T	T	T
F	T	F	F	F	F
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_{13}(p, q)$	$op_{14}(p, q)$	$op_{15}(p, q)$	$op_{16}(p, q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	T	F	T

Three Famous Ones

op_2 : $op_2(p, q)$ is T when p is T *and* q is T, and F otherwise. Called *conjunction*, denoted $p \wedge q$.

op_8 : $op_8(p, q)$ is T when p is T *or* q is T, and F otherwise. Called *disjunction*, denoted $p \vee q$.

op_{14} : $op_{14}(p, q)$ is T when p is F *or* q is T, and F otherwise. Called *implication*, denoted $p \rightarrow q$.

Inductive Definition

Definition

For a given set A of propositional atoms, the set of *well-formed formulas in propositional logic* is the smallest set F that fulfills the following rules:

- The constant symbols $\perp$ and $\top$ are in F.
- Every element of A is in F.
- If ϕ is in F, then $(\neg\phi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \wedge \psi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \vee \psi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \rightarrow \psi)$ is also in F.

Notes on Defining Sets

Are there more sets that fulfill these rules?

The set of *all formula defined in this module* fulfills these rules. The clause “the smallest set” eliminates those elements that are not *forced in* through any of the given rules.

Observe

Some rules are simple (they do not have any condition), whereas others assume one or two formulas in the set as given, and then require that another forumula is also in the set.

Inductively defined sets

A set defined in this way is called an *inductively defined set*.

Example

$$(((\neg p) \wedge q) \rightarrow (\top \wedge (q \vee (\neg r))))$$

is a well-formed formula in propositional logic.

More Compact in Backus-Naur-Form (BNF)

$$\phi ::= p \mid \bot \mid \top \mid (\neg\phi) \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \rightarrow \phi)$$

Convention

The negation symbol $\neg$ binds more tightly than $\wedge$ and $\vee$, and $\wedge$ and $\vee$ bind more tightly than $\rightarrow$. Moreover, $\rightarrow$ is *right-associative*: The formula $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$.

Example

$$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$$

can be written as

$$\neg p \wedge q \rightarrow p \wedge (q \vee \neg r)$$

Negating Truth Values

Definition

Function $\backslash : \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B	$\backslash B$
F	T
T	F

Conjunction of Truth Values

Definition

Function $\& : \{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \& B_2$
F	F	F
F	T	F
T	F	F
T	T	T

Disjunction of Truth Values

Definition

Function $|$: $\{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \mid B_2$
F	F	F
F	T	T
T	F	T
T	T	T

Implication of Truth Values

Definition

Function $\Rightarrow$: $\{F, T\} \times \{F, T\} \to \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \Rightarrow B_2$
F	F	T
F	T	T
T	F	F
T	T	T

Evaluation of Formulas

Definition

The result of *evaluating* a well-formed propositional formula ϕ with respect to a valuation v, denoted $v(\phi)$ is defined as follows:

- If ϕ is the constant $\perp$, then $v(\phi) = F$.
- If ϕ is the constant $\top$, then $v(\phi) = T$.
- If ϕ is an propositional atom p, then $v(\phi) = p^v$.
- If ϕ has the form $(\neg\psi)$, then $v(\phi) = \backslash v(\psi)$.
- If ϕ has the form $(\psi \wedge \tau)$, then $v(\phi) = v(\psi) \& v(\tau)$.
- If ϕ has the form $(\psi \vee \tau)$, then $v(\phi) = v(\psi) \mid v(\tau)$.
- If ϕ has the form $(\psi \rightarrow \tau)$, then $v(\phi) = v(\psi) \Rightarrow v(\tau)$.

Valid Formulas

Definition

A formula is called *valid* if it evaluates to T with respect to every possible valuation.

Examples

Example

Is

$$(((\neg p) \wedge q) \rightarrow (\top \wedge (q \vee (\neg r))))$$

valid?

Example

Find a valid formula that contains the propositional atoms p, q, r and w.

Meaning of propositional formula

Meaning as mathematical object

We define the meaning of formulas as a function that maps formulas and valuations to truth values.

Approach

We define this mapping based on the structure of the formula, using the meaning of their logical connectives.

Truth values and valuations

Definition

The set of truth values contains two elements $\mathtt{T}$ and $\mathtt{F}$, where $\mathtt{T}$ represents "true" and $\mathtt{F}$ represents "false".

Definition

A *valuation* or *model* of a formula ϕ is an assignment of each propositional atom in ϕ to a truth value.

Meaning of logical connectives

The meaning of a connective is defined as a truth table that gives the truth value of a formula, whose root symbol is the connective, based on the truth values of its components.

ϕ	ψ	$\phi \land \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Truth tables of formulas

Truth tables use placeholders of formulas such as ϕ:

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Build the truth table for given formula:

p	q	r	$(p \wedge q)$	$((p \wedge q) \wedge r)$
T	T	T	T	T
T	T	F	T	F
$\vdots$				

Truth tables of other connectives

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

ϕ	ψ	$\phi \to \psi$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	$\neg\phi$
T	F
F	T

$\top$
T

$\bot$
F

Constructing the truth table of a formula

p	q	$(\neg p)$	$\neg q$	$p \rightarrow \neg q$	$q \vee \neg p$	$(p \rightarrow \neg q) \rightarrow (q \vee \neg p)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Validity and Satisfiability

Validity

A formula is *valid* if it computes $\mathtt{T}$ for all its valuations.

Satisfiability

A formula is *satisfiable* if it computes $\mathtt{T}$ for at least one of its valuations.

1. Atoms and Propositions
2. Semantics of Propositional Logic
3. Outlook: Natural Deduction for Propositional Logic
   - Sequents

Introduction

Objective

We would like to develop a *calculus* for reasoning about propositions, so that we can establish the validity of statements such as “If the train arrives late...”.

Idea

We introduce *proof rules* that allow us to derive a formula ψ from a number of other formulas $\phi_1, \phi_2, \ldots \phi_n$.

Notation

We write a *sequent* $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$
to denote that we can derive ψ from $\phi_1, \phi_2, \ldots, \phi_n$.

Example—Revisited

English

If *the train arrives late* and *there are no taxis at the station*, then *John is late for his meeting.*

John is not late for his meeting. The train did arrive late.

Therefore, *there were taxis at the station.*

Sequent

$$p \wedge \neg q \rightarrow r, \neg r, p \vdash q$$

Remaining task

Find proof rules that allow us to establish such sequents.