06—From Propositional to Predicate Logic

The Importance of Being Formal

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1. Syntax of Predicate Logic
   - Need for Richer Language
   - Predicates
   - Variables
   - Functions
2. Predicate Logic as a Formal Language
3. Semantics of Predicate Logic
4. Proof Theory
5. Equivalences

More Declarative Sentences

- Propositional logic can easily handle simple declarative statements such as:

Example

Student Peter Lim enrolled in UIT2206.

- Propositional logic can also handle combinations of such statements such as:

Example

Student Peter Lim enrolled in Tutorial 1, *and* student Julie Bradshaw is enrolled in Tutorial 2.

- *But:* How about statements with *"there exists..."* or *"every..."* or *"among..."*?

What is needed?

Example

Every student is younger than *some* instructor.

What is this statement about?

- Being a student
- Being an instructor
- Being younger than somebody else

These are *properties* of elements of a *set* of objects.

We express them in predicate logic using *predicates*.

Predicates

Example

Every student is younger than *some* instructor.

- $S(andy)$ could denote that Andy is a student.
- $I(paul)$ could denote that Paul is an instructor.
- $Y(andy, paul)$ could denote that Andy is younger than Paul.

The Need for Variables

Example

Every student is younger than *some* instructor.

We use the predicate S to denote student-hood.
How do we express *"every student"*?

We need *variables* that can stand for constant values, and a *quantifier* symbol that denotes *"every"*.

The Need for Variables

Example

Every student is younger than *some* instructor.

Using variables and quantifiers, we can write:

$$\forall x(S(x) \rightarrow (\exists y(I(y) \wedge Y(x, y)))).$$

Literally: For every x, if x is a student, then there is some y such that y is an instructor and x is younger than y.

Another Example

English

Not all birds can fly.

Predicates

$B(x)$: x is a bird

$F(x)$: x can fly

The sentence in predicate logic

$$\neg(\forall x(B(x) \rightarrow F(x)))$$

A Third Example

English

Every girl is younger than her mother.

Predicates

$G(x)$: x is a girl

$M(x, y)$: x is y's mother

$Y(x, y)$: x is younger than y

The sentence in predicate logic

$$\forall x \forall y (G(x) \land M(y, x) \rightarrow Y(x, y))$$

A “Mother” Function

The sentence in predicate logic

$$\forall x \forall y (G(x) \land M(y, x) \rightarrow Y(x, y))$$

Note that y is only introduced to denote the mother of x.

If everyone has exactly one mother, the predicate $M(y, x)$ is a function, when read from right to left.

We introduce a function symbol m that can be applied to variables and constants as in

$$\forall x (G(x) \rightarrow Y(x, m(x)))$$

A Drastic Example

English

Andy and Paul have the same maternal grandmother.

The sentence in predicate logic without functions

$$\forall x \forall y \forall u \forall v (M(x, y) \land M(y, andy) \land$$
$$M(u, v) \land M(v, paul) \rightarrow x = u)$$

The same sentence in predicate logic with functions

$$m(m(andy)) = m(m(paul))$$

Outlook

Syntax: We formalize the language of predicate logic, including scoping and substitution.

Semantics: We describe models in which predicates, functions, and formulas have meaning.

Proof theory: We extend natural deduction from propositional to predicate logic

Further topics: Soundness/completeness, undecidability, incompleteness results, compactness results

Predicate Vocabulary

At any point in time, we want to describe the features of a particular "world", using predicates, functions, and constants. Thus, we introduce for this world:

- a set of predicate symbols $\mathcal{P}$
- a set of function symbols $\mathcal{F}$

Arity of Functions and Predicates

Every function symbol in $\mathcal{F}$ and predicate symbol in $\mathcal{P}$ comes with a fixed arity, denoting the number of arguments the symbol can take.

Special case: Nullary Functions

Function symbols with arity 0 are called *constants*.

Special case: Nullary Predicates

Predicate symbols with arity 0 denotes predicates that do not depend on any arguments. They correspond to propositional atoms.

Terms

$$t ::= \; x \mid c \mid f(t, \ldots, t)$$

where

- x ranges over a given set of variables $\mathcal{V}$,
- c ranges over nullary function symbols in $\mathcal{F}$, and
- f ranges over function symbols in $\mathcal{F}$ with arity $n > 0$.

Examples of Terms

If n is nullary, f is unary, and g is binary, then examples of terms are:

- $g(f(n), n)$
- $f(g(n, f(n)))$

More Examples of Terms

If $0, 1, 2$ are nullary (constants), s is unary, and $+, -$ and $*$ are binary, then

$$*(-(2, +(s(x), y)), x)$$

is a term.
Occasionally, we allow ourselves to use infix notation for function symbols as in

$$(2 - (s(x) + y)) * x$$

Formulas

$$\begin{aligned}\phi \quad ::= \quad & P(t, \ldots, t) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \\ & (\phi \rightarrow \phi) \mid (\forall x \phi) \mid (\exists x \phi)\end{aligned}$$

where

- $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 0$,
- t are terms over $\mathcal{F}$ and $\mathcal{V}$, and
- x are variables in $\mathcal{V}$.

Conventions

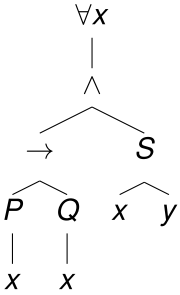
Just like for propositional logic, we introduce convenient conventions to reduce the number of parentheses:

- $\neg, \forall x$ and $\exists x$ bind most tightly;
- then $\wedge$ and $\vee$;
- then $\rightarrow$, which is right-associative.

Parse Trees

$$\forall x((P(x) \rightarrow Q(x)) \wedge S(x, y))$$

has parse tree



Another Example

Every son of my father is my brother.

Predicates

$S(x, y)$: x is a son of y

$B(x, y)$: x is a brother of y

Functions

m: constant for "me"

$f(x)$: father of x

The sentence in predicate logic

$$\forall x(S(x, f(m)) \to B(x, m))$$

Equality as Predicate

Equality is a common predicate, usually used in infix notation.

$$= \in \mathcal{P}$$

Example

Instead of the formula

$$=(f(x), g(x))$$

we usually write the formula

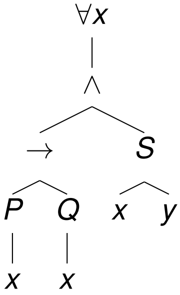
$$f(x) = g(x)$$

Free and Bound Variables

Consider the formula

$$\forall x((P(x) \to Q(x)) \wedge S(x, y))$$

What is the relationship between variable “binder” x and occurrences of x?

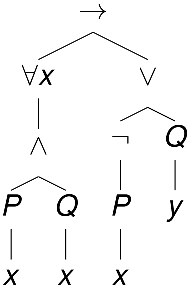


Free and Bound Variables

Consider the formula

$$(\forall x(P(x) \land Q(x))) \to (\neg P(x) \lor Q(y))$$

Which variable *occurrences* are free; which are bound?



Substitution

Variables are *place*holders. Re*plac*ing them by terms is called *substitution*.

Definition

Given a variable x, a term t and a formula ϕ, we define $[x \Rightarrow t]\phi$ to be the formula obtained by replacing each free occurrence of variable x in ϕ with t.

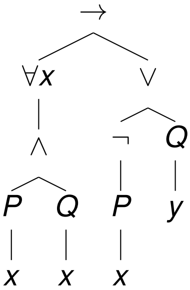
Example

$$[x \Rightarrow f(x,y)]((\forall x(P(x) \land Q(x))) \to (\neg P(x) \lor Q(y)))$$
$$= \forall x(P(x) \land Q(x))) \to (\neg P(f(x,y)) \lor Q(y))$$

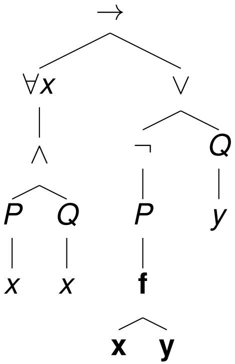
Example as Parse Tree

$$[x \Rightarrow f(x, y)]((\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y)))$$

$$= (\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(f(x, y)) \vee Q(y))$$



Example as Parse Tree



Models

Definition

Let $\mathcal{F}$ contain function symbols and $\mathcal{P}$ contain predicate symbols. A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A non-empty set A, the *universe*;
2. for each nullary function symbol $f \in \mathcal{F}$ a concrete element $f^{\mathcal{M}} \in A$;
3. for each $f \in F$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$;
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a function $P^{\mathcal{M}} : U^n \to \{F, T\}$.
5. for each $P \in \mathcal{P}$ with arity $n = 0$, a value from $\{F, T\}$.

Example

Let $\mathcal{F} = \{e, \cdot\}$ and $\mathcal{P} = \{\leq\}$.
Let model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ be defined as follows:

1. Let A be the set of binary strings over the alphabet $\{0, 1\}$;
2. let $e^{\mathcal{M}} = \epsilon$, the empty string;
3. let $\cdot^{\mathcal{M}}$ be defined such that $s_1 \cdot^{\mathcal{M}} s_2$ is the concatenation of the strings s_1 and s_2; and
4. let $\leq^{\mathcal{M}}$ be defined such that $s_1 \leq^{\mathcal{M}} s_2$ iff s_1 is a prefix of s_2.

Example (continued)

1. Let A be the set of binary strings over the alphabet $\{0, 1\}$;
2. let $e^{\mathcal{M}} = \epsilon$, the empty string;
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4. let $\leq^{\mathcal{M}}$ be defined such that $s_1 \leq^{\mathcal{M}} s_2$ iff s_1 is a prefix of s_2.

Some Elements of A

- 10001
- ϵ
- $1010 \cdot^{\mathcal{M}} 1100 = 10101100$
- $000 \cdot^{\mathcal{M}} \epsilon = 000$

Equality Revisited

Interpretation of equality

Usually, we require that the equality predicate $=$ is interpreted as same-ness.

Extensionality restriction

This means that allowable models are restricted to those in which $a =^{\mathcal{M}} b$ holds if and only if a and b are the same elements of the model's universe.

Example (continued)

1. Let A be the set of binary strings over the alphabet $\{0, 1\}$;
2. let $e^{\mathcal{M}} = \epsilon$, the empty string;
3. let $\cdot^{\mathcal{M}}$ be defined such that $s_1 \cdot^{\mathcal{M}} s_2$ is the concatenation of the strings s_1 and s_2; and
4. let $\leq^{\mathcal{M}}$ be defined such that $s_1 \leq^{\mathcal{M}} s_2$ iff s_1 is a prefix of s_2.

Equality in $\mathcal{M}$

- $000 =^{\mathcal{M}} 000$
- $001 \neq^{\mathcal{M}} 100$

Another Example

Let $\mathcal{F} = \{z, s\}$ and $\mathcal{P} = \{\leq\}$.
Let model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ be defined as follows:

1. Let A be the set of natural numbers;
2. let $z^{\mathcal{M}} = 0$;
3. let $s^{\mathcal{M}}$ be defined such that $s(n) = n + 1$; and
4. let $\leq^{\mathcal{M}}$ be defined such that $n_1 \leq^{\mathcal{M}} n_2$ iff the natural number n_1 is less than or equal to n_2.

How To Handle Free Variables?

Idea

We can give meaning to formulas with free variables by providing an environment (lookup table) that assigns variables to elements of our universe:

$$I : \mathcal{V} \to A.$$

Environment extension

We define environment extension such that $I[x \mapsto a]$ is the environment that maps x to a and any other variable y to $I(y)$.

Satisfaction Relation

The model $\mathcal{M}$ satisfies ϕ with respect to environment l, written $\mathcal{M} \models_l \phi$:

- in case ϕ is of the form $P(t_1, t_2, \ldots, t_n)$, if $a_1, a_2, \ldots, a_n$ are the results of evaluating $t_1, t_2, \ldots, t_n$ with respect to l, and if $P^{\mathcal{M}}(a_1, a_2, \ldots, a_n) = T$;
- in case ϕ is of the form P, if $P^{\mathcal{M}} = T$;
- in case ϕ has the form $\forall x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for all $a \in A$;
- in case ϕ has the form $\exists x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for some $a \in A$;

Satisfaction Relation (continued)

- in case ϕ has the form $\neg\psi$, if $\mathcal{M} \models_l \psi$ does not hold;
- in case ϕ has the form $\psi_1 \vee \psi_2$, if $\mathcal{M} \models_l \psi_1$ holds or $\mathcal{M} \models_l \psi_2$ holds;
- in case ϕ has the form $\psi_1 \wedge \psi_2$, if $\mathcal{M} \models_l \psi_1$ holds and $\mathcal{M} \models_l \psi_2$ holds; and
- in case ϕ has the form $\psi_1 \rightarrow \psi_2$, if $\mathcal{M} \models_l \psi_2$ holds whenever $\mathcal{M} \models_l \psi_1$ holds.

Satisfaction of Closed Formulas

If a formula ϕ has no free variables, we call ϕ a *sentence*.
$\mathcal{M} \models_I \phi$ holds or does not hold regardless of the choice of I.
Thus we write $\mathcal{M} \models \phi$ or $\mathcal{M} \not\models \phi$.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

Entailment

$\Gamma \models \psi$ iff for all models $\mathcal{M}$ and environments l, whenever $\mathcal{M} \models_l \phi$ holds for all $\phi \in \Gamma$, then $\mathcal{M} \models_l \psi$.

Satisfiability of Formulas

ψ is satisfiable iff there is some model $\mathcal{M}$ and some environment l such that $\mathcal{M} \models_l \psi$ holds.

Satisfiability of Formula Sets

Γ is satisfiable iff there is some model $\mathcal{M}$ and some environment l such that $\mathcal{M} \models_l \phi$, for all $\phi \in \Gamma$.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

Validity

ψ is valid iff for all models $\mathcal{M}$ and environments l, we have $\mathcal{M} \models_l \psi$.

The Problem with Predicate Logic

Entailment ranges over models

Semantic entailment between sentences: $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ requires that in *all* models that satisfy $\phi_1, \phi_2, \ldots, \phi_n$, the sentence ψ is satisfied.

How to effectively argue about all possible models?

Usually the number of models is infinite; it is very hard to argue on the semantic level in predicate logic.

Idea from propositional logic

Can we use natural deduction for showing entailment?

Natural Deduction for Predicate Logic

Relationship between propositional and predicate logic

If we consider propositions as nullary predicates, propositional logic is a sub-language of predicate logic.

Inheriting natural deduction

We can translate the rules for natural deduction in propositional logic directly to predicate logic.

Example

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}[\wedge i]$$

Built-in Rules for Equality

$$\frac{}{t = t}[= i] \qquad \frac{t_1 = t_2 \qquad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi}[= e]$$

Properties of Equality

We show:

$$f(x) = g(x) \vdash h(g(x)) = h(f(x))$$

using

$$\frac{}{t = t}[= i] \qquad \frac{t_1 = t_2 \qquad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi}[= e]$$

1	$f(x) = g(x)$	premise
2	$h(f(x)) = h(f(x))$	$= i$
3	$h(g(x)) = h(f(x))$	$= e$ 1,2

Elimination of Universal Quantification

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi}[\forall x\ e]$$

Once you have proven $\forall x \phi$, you can replace x by any term t in ϕ, provided that t is free for x in ϕ.

Example

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi}[\forall x\ e]$$

We prove: $S(g(john)), \forall x(S(x) \rightarrow \neg L(x)) \vdash \neg L(g(john))$

1	$S(g(john))$	premise
2	$\forall x(S(x) \rightarrow \neg L(x))$	premise
3	$S(g(john)) \rightarrow \neg L(g(john))$	$\forall x\ e$ 2
4	$\neg L(g(john))$	$\rightarrow e$ 3,1

Introduction of Universal Quantification

$$\frac{\boxed{\begin{array}{c}\vdots\\ [x \Rightarrow x_0]\phi\end{array}}^{\,x_0}}{\forall x \phi}\;[\forall x\ i]$$

If we manage to establish a formula ϕ about a fresh variable x_0, we can assume $\forall x \phi$.

The variable x_0 must be *fresh*; we cannot introduce the same variable twice in nested boxes.

Example

$\forall x(P(x) \rightarrow Q(x)), \forall x P(x) \vdash \forall x Q(x)$ via

$$\frac{\boxed{\begin{array}{c} \vdots \\ [x \Rightarrow x_0]\phi \end{array}}^{\,x_0}}{\forall x \phi}$$

1	$\forall x(P(x) \rightarrow Q(x))$	premise	
2	$\forall x P(x)$	premise	
3	$P(x_0) \rightarrow Q(x_0)$	$\forall x\ e$ 1	x_0
4	$P(x_0)$	$\forall x\ e$ 2	
5	$Q(x_0)$	$\rightarrow e$ 3,4	
6	$\forall x Q(x)$	$\forall x\ i$ 3–5	

Introduction of Existential Quantification

$$\frac{[x \Rightarrow t]\phi}{\exists x \phi}\,[\exists x\ i]$$

In order to prove $\exists x\phi$, it suffices to find a term t as “witness”, provided that t is free for x in ϕ.

Example

$$\forall x \phi \vdash \exists x \phi$$

Recall: Definition of Models

A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A *non-empty* set U, the *universe*;
2. ...

Remark

Compare this with Traditional Logic.

Because U must not be empty, we should be able to prove the sequent above.

Example (continued)

$$\forall x\phi \vdash \exists x\phi$$

1	$\forall x\phi$	premise
2	$[x \Rightarrow x]\phi$	$\forall x\ e\ 1$
3	$\exists x\phi$	$\exists x\ i\ 2$

Elimination of Existential Quantification

$$
\dfrac{\exists x \phi \qquad \boxed{\begin{array}{c} [x \Rightarrow x_0]\phi \\ \vdots \\ \chi \end{array}}^{\,x_0}_{\,[x\Rightarrow x_0]\phi}}{\chi}\;[\exists e]
$$

Making use of $\exists$

If we know $\exists x \phi$, we know that there exist at least one object x for which ϕ holds. We call that element x_0, and assume

Example

$$\forall x(P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$$

1	$\forall x(P(x) \rightarrow Q(x))$	premise	
2	$\exists x P(x)$	premise	
3	$P(x_0)$	assumption	x_0
4	$P(x_0) \rightarrow Q(x_0)$	$\forall x\ e$ 1	
5	$Q(x_0)$	$\rightarrow e$ 4,3	
6	$\exists x Q(x)$	$\exists x\ i$ 5	
7	$\exists x Q(x)$	$\exists x\ e$ 2,3–6	

Note that $\exists x Q(x)$ within the box does not contain x_0, and therefore can be “exported” from the box.

Another Example

1	$\forall x(Q(x) \to R(x))$	premise	
2	$\exists x(P(x) \land Q(x))$	premise	
3	$P(x_0) \land Q(x_0)$	assumption	x_0
4	$Q(x_0) \to R(x_0)$	$\forall x\ e$ 1	
5	$Q(x_0)$	$\land e_2$ 3	
6	$R(x_0)$	$\to e$ 4,5	
7	$P(x_0)$	$\land e_1$ 3	
8	$P(x_0) \land R(x_0)$	$\land i$ 7, 6	
9	$\exists x(P(x) \land R(x)$	$\exists x\ i$ 8	
10	$\exists x(P(x) \land R(x))$	$\exists x\ e$ 2,3–9	

Variables must be fresh! This is not a proof!

1	$\exists x P(x)$	premise	
2	$\forall x (P(x) \to Q(x))$	premise	
3			x_0
4	$P(x_0)$	assumption	x_0
5	$P(x_0) \to Q(x_0)$	$\forall x\ e$ 2	
6	$Q(x_0)$	$\to e$ 5,4	
7	$Q(x_0)$	$\exists x\ e$ 1, 4–6	
8	$\forall y Q(y)$	$\forall y\ i$ 3–7	

Equivalences

Two-way-provable

We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Some simple equivalences

$$
\begin{array}{rcl}
\neg\forall x\phi & \dashv\vdash & \exists x\neg\phi \\
\neg\exists x\phi & \dashv\vdash & \forall x\neg\phi \\
\forall x\forall y\phi & \dashv\vdash & \forall y\forall x\phi \\
\exists x\exists y\phi & \dashv\vdash & \exists y\exists x\phi \\
\forall x\phi \wedge \forall x\psi & \dashv\vdash & \forall x(\phi \wedge \psi) \\
\exists x\phi \vee \exists x\psi & \dashv\vdash & \exists x(\phi \vee \psi)
\end{array}
$$

$\neg\forall x\phi \vdash \exists x\neg\phi$

1	$\neg\forall x\phi$	premise
2	$\neg\exists x\neg\phi$	assumption
3		x_0
4	$\neg[x \Rightarrow x_0]\phi$	assumption
5	$\exists x\neg\phi$	$\exists x\ i$ 4
6	$\bot$	$\neg e$ 5, 2
7	$[x \Rightarrow x_0]\phi$	PBC 4–6
8	$\forall x\phi$	$\forall x\ i$ 3–7
9	$\bot$	$\neg e$ 8, 1
10	$\exists x\neg\phi$	PBC 2–9

$\exists x \exists y \phi \vdash \exists y \exists x \phi$

Assume that x and y are different variables.

1	$\exists x \exists y \phi$	premise	
2	$[x \Rightarrow x_0](\exists y \phi)$	assumption	x_0
3	$\exists y([x \Rightarrow x_0]\phi$	def of subst (x, y different)	
4	$[y \Rightarrow y_0][x \Rightarrow x_0]\phi$	assumption	y_0
5	$[x \Rightarrow x_0][y \Rightarrow y_0]\phi$	def of subst (x, y, x_0, y_0 different)	
6	$\exists x[y \rightarrow y_0]\phi$	$\exists x$ i 5	
7	$\exists y \exists x \phi$	$\exists y$ i 6	
8	$\exists y \exists x \phi$	$\exists y$ e 3, 4–7	
9	$\exists y \exists x \phi$	$\exists x$ e 1, 2–8	

More Equivalences

Assume that x is not free in ψ

$$
\begin{aligned}
\forall x\phi \wedge \psi &\quad \dashv\vdash \quad \forall x(\phi \wedge \psi) \\
\forall x\phi \vee \psi &\quad \dashv\vdash \quad \forall x(\phi \vee \psi) \\
\exists x\phi \wedge \psi &\quad \dashv\vdash \quad \exists x(\phi \wedge \psi) \\
\exists x\phi \vee \psi &\quad \dashv\vdash \quad \exists x(\phi \vee \psi)
\end{aligned}
$$

Central Result of Natural Deduction

$$\phi_1, \ldots, \phi_n \models \psi$$

iff

$$\phi_1, \ldots, \phi_n \vdash \psi$$

proven by Kurt Gödel, in 1929 in his doctoral dissertation (just one year before his most famous result, the incompleteness results of mathematical logic)