The Importance of Being Formal

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1 Syntax of Predicate Logic
2 Predicate Logic as a Formal Language
3 Semantics of Predicate Logic
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5 Equivalences
6 Soundness and Completeness
Propositional logic can easily handle simple declarative statements such as:

Example

Student Peter Lim enrolled in UIT2206.

Propositional logic can also handle combinations of such statements such as:

Example

Student Peter Lim enrolled in Tutorial 1, \textit{and} student Julie Bradshaw is enrolled in Tutorial 2.

\textit{But}: How about statements with \textit{“there exists...”} or \textit{“every...”} or \textit{“among...”?}
What is needed?

Example

Every student is younger than some instructor.

What is this statement about?

- Being a student
- Being an instructor
- Being younger than somebody else

These are properties of elements of a set of objects.

We express them in predicate logic using predicates.
Predicates

Example

Every student is younger than some instructor.

- $S(andy)$ could denote that Andy is a student.
- $I(paul)$ could denote that Paul is an instructor.
- $Y(andy, paul)$ could denote that Andy is younger than Paul.
The Need for Variables

Example

Every student is younger than some instructor.

We use the predicate $S$ to denote student-hood. How do we express “every student”?

We need variables that can stand for constant values, and a quantifier symbol that denotes “every”.

The Importance of Being Formal
The Need for Variables

Example

Every student is younger than some instructor.

Using variables and quantifiers, we can write:

$$\forall x (S(x) \rightarrow (\exists y (I(y) \land Y(x, y))))$$

Literally: For every $x$, if $x$ is a student, then there is some $y$ such that $y$ is an instructor and $x$ is younger than $y$. 

Another Example

**English**
Not all birds can fly.

**Predicates**
- $B(x)$: $x$ is a bird
- $F(x)$: $x$ can fly

**The sentence in predicate logic**

$$\neg(\forall x(B(x) \rightarrow F(x)))$$
A Third Example

English

Every girl is younger than her mother.

Predicates

- $G(x)$: $x$ is a girl
- $M(x, y)$: $x$ is $y$’s mother
- $Y(x, y)$: $x$ is younger than $y$

The sentence in predicate logic

$$\forall x \forall y (G(x) \land M(y, x) \rightarrow Y(x, y))$$
A “Mother” Function

The sentence in predicate logic

$$\forall x \forall y (G(x) \land M(y, x) \rightarrow Y(x, y))$$

Note that $y$ is only introduced to denote the mother of $x$.

If everyone has exactly one mother, the predicate $M(y, x)$ is a function, when read from right to left.

We introduce a function symbol $m$ that can be applied to variables and constants as in

$$\forall x (G(x) \rightarrow Y(x, m(x)))$$
A Drastic Example

English
Andy and Paul have the same maternal grandmother.

The sentence in predicate logic without functions

\[ \forall x \forall y \forall u \forall v (M(x, y) \land M(y, andy) \land M(u, v) \land M(v, paul) \rightarrow x = u) \]

The same sentence in predicate logic with functions

\[ m(m(andy)) = m(m(paul)) \]
Syntax: We formalize the language of predicate logic, including scoping and substitution.

Semantics: We describe models in which predicates, functions, and formulas have meaning.

Proof theory: We extend natural deduction from propositional to predicate logic.

Further topics: Soundness/completeness, undecidability, incompleteness results, compactness results.
Predicate Vocabulary

At any point in time, we want to describe the features of a particular “world”, using predicates, functions, and constants. Thus, we introduce for this world:

- a set of predicate symbols $\mathcal{P}$
- a set of function symbols $\mathcal{F}$
Arity of Functions and Predicates

Every function symbol in $\mathcal{F}$ and predicate symbol in $\mathcal{P}$ comes with a fixed arity, denoting the number of arguments the symbol can take.

**Special case: Nullary Functions**

Function symbols with arity 0 are called *constants*.

**Special case: Nullary Predicates**

Predicate symbols with arity 0 denotes predicates that do not depend on any arguments. They correspond to propositional atoms.
Terms

\[ t ::= x \mid c \mid f(t, \ldots, t) \]

where

- \( x \) ranges over a given set of variables \( \mathcal{V} \),
- \( c \) ranges over nullary function symbols in \( \mathcal{F} \), and
- \( f \) ranges over function symbols in \( \mathcal{F} \) with arity \( n > 0 \).
Examples of Terms

If $n$ is nullary, $f$ is unary, and $g$ is binary, then examples of terms are:

- $g(f(n), n)$
- $f(g(n, f(n)))$
More Examples of Terms

If 0, 1, 2 are nullary (constants), s is unary, and +, − and ∗ are binary, then

\[ ∗(−(2, +(s(x), y)), x) \]

is a term.

Occasionally, we allow ourselves to use infix notation for function symbols as in

\[ (2 − (s(x) + y)) ∗ x \]
Formulas

\[ \phi ::= P(t, \ldots, t) | (\neg \phi) | (\phi \land \phi) | (\phi \lor \phi) | \\
(\phi \rightarrow \phi) | (\forall x\phi) | (\exists x\phi) \]

where

- \( P \in \mathcal{P} \) is a predicate symbol of arity \( n \geq 0 \),
- \( t \) are terms over \( \mathcal{F} \) and \( \mathcal{V} \), and
- \( x \) are variables in \( \mathcal{V} \).
Conventions

Just like for propositional logic, we introduce convenient conventions to reduce the number of parentheses:

- $\neg$, $\forall x$ and $\exists x$ bind most tightly;
- then $\land$ and $\lor$;
- then $\rightarrow$, which is right-associative.
The Importance of Being Formal

The diagram illustrates the parse tree of the formula:

$$\forall x((P(x) \to Q(x)) \land S(x, y))$$

The parse tree shows the breakdown of the formula into its constituent parts, with $\forall x$, $\to$, $\land$, $S$, $P$, $Q$, $x$, and $y$ as nodes.
Another Example

Every son of my father is my brother.

Predicates

\[ S(x, y) : \text{x is a son of y} \]
\[ B(x, y) : \text{x is a brother of y} \]

Functions

\[ m : \text{constant for “me”} \]
\[ f(x) : \text{father of x} \]

The sentence in predicate logic

\[ \forall x (S(x, f(m)) \rightarrow B(x, m)) \]
Equality as Predicate

Equality is a common predicate, usually used in infix notation.

\[ = \in \mathcal{P} \]

Example

Instead of the formula

\[ = (f(x), g(x)) \]

we usually write the formula

\[ f(x) = g(x) \]
Consider the formula

$$\forall x((P(x) \rightarrow Q(x)) \land S(x, y))$$

What is the relationship between variable “binder” $x$ and occurrences of $x$?
Consider the formula

$$(\forall x(P(x) \land Q(x))) \rightarrow (\neg P(x) \lor Q(y))$$

Which variable occurrences are free; which are bound?
Variables are place holders. Replacing them by terms is called substitution.

**Definition**

Given a variable $x$, a term $t$ and a formula $\phi$, we define $[x \Rightarrow t]\phi$ to be the formula obtained by replacing each free occurrence of variable $x$ in $\phi$ with $t$.

**Example**

$$[x \Rightarrow f(x, y)]((\forall x(P(x) \land Q(x))) \rightarrow (\neg P(x) \lor Q(y)))$$

$$= \forall x(P(x) \land Q(x))) \rightarrow (\neg P(f(x, y)) \lor Q(y))$$
Example as Parse Tree

\[\left[ x \Rightarrow f(x, y) \right] \left( (\forall x (P(x) \land Q(x))) \rightarrow (\neg P(x) \lor Q(y)) \right) \]

\[= (\forall x (P(x) \land Q(x))) \rightarrow (\neg P(f(x, y)) \lor Q(y)) \]

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06—From Propositional to Predicate Logic
Example as Parse Tree

\[ \forall x (P \land Q) \lor \neg P(y) \]

The Importance of Being Formal
1. Syntax of Predicate Logic

2. Predicate Logic as a Formal Language

3. Semantics of Predicate Logic
   - Models
   - Equality
   - Free Variables
   - Satisfaction and Entailment

4. Proof Theory

5. Equivalences
Definition

Let $\mathcal{F}$ contain function symbols and $\mathcal{P}$ contain predicate symbols. A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A non-empty set $A$, the *universe*;
2. for each nullary function symbol $f \in \mathcal{F}$ a concrete element $f^\mathcal{M} \in A$;
3. for each $f \in F$ with arity $n > 0$, a concrete function $f^\mathcal{M} : A^n \rightarrow A$;
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a function $P^\mathcal{M} : U^n \rightarrow \{F, T\}$.
5. for each $P \in \mathcal{P}$ with arity $n = 0$, a value from $\{F, T\}$. 

The Importance of Being Formal

06—From Propositional to Predicate Logic
Let $\mathcal{F} = \{e, \cdot\}$ and $\mathcal{P} = \{\leq\}$. Let model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ be defined as follows:

1. Let $A$ be the set of binary strings over the alphabet $\{0, 1\}$;
2. let $e^\mathcal{M} = \epsilon$, the empty string;
3. let $\cdot^\mathcal{M}$ be defined such that $s_1 \cdot^\mathcal{M} s_2$ is the concatenation of the strings $s_1$ and $s_2$; and
4. let $\leq^\mathcal{M}$ be defined such that $s_1 \leq^\mathcal{M} s_2$ iff $s_1$ is a prefix of $s_2$. 
Example (continued)

1. Let $A$ be the set of binary strings over the alphabet $\{0, 1\}$;
2. let $e^M = \epsilon$, the empty string;
3. let $\cdot^M$ be defined such that $s_1 \cdot^M s_2$ is the concatenation of the strings $s_1$ and $s_2$; and
4. let $\leq^M$ be defined such that $s_1 \leq^M s_2$ iff $s_1$ is a prefix of $s_2$.

Some Elements of $A$

- 10001
- $\epsilon$
- $1010 \cdot^M 1100 = 10101100$
- $000 \cdot^M \epsilon = 000$
Equality Revisited

Interpretation of equality

Usually, we require that the equality predicate $=$ is interpreted as same-ness.

Extensionality restriction

This means that allowable models are restricted to those in which $a =^M b$ holds if and only if $a$ and $b$ are the same elements of the model’s universe.
Example (continued)

1. Let $A$ be the set of binary strings over the alphabet $\{0, 1\}$;
2. let $e^M = \epsilon$, the empty string;
3. let $\cdot^M$ be defined such that $s_1 \cdot^M s_2$ is the concatenation of the strings $s_1$ and $s_2$; and
4. let $\leq^M$ be defined such that $s_1 \leq^M s_2$ iff $s_1$ is a prefix of $s_2$.

Equality in $M$

- $000 =^M 000$
- $001 \neq^M 100$
Let $\mathcal{F} = \{z, s\}$ and $\mathcal{P} = \{\leq\}$. Let model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ be defined as follows:

1. Let $A$ be the set of natural numbers;
2. let $z^\mathcal{M} = 0$;
3. let $s^\mathcal{M}$ be defined such that $s(n) = n + 1$; and
4. let $\leq^\mathcal{M}$ be defined such that $n_1 \leq^\mathcal{M} n_2$ iff the natural number $n_1$ is less than or equal to $n_2$. 
Idea
We can give meaning to formulas with free variables by providing an environment (lookup table) that assigns variables to elements of our universe:

\[ l : \forall \rightarrow A. \]

Environment extension
We define environment extension such that \( l[x \mapsto a] \) is the environment that maps \( x \) to \( a \) and any other variable \( y \) to \( l(y) \).
The model $\mathcal{M}$ satisfies $\phi$ with respect to environment $l$, written $\mathcal{M} \models_l \phi$:

- in case $\phi$ is of the form $P(t_1, t_2, \ldots, t_n)$, if $a_1, a_2, \ldots, a_n$ are the results of evaluating $t_1, t_2, \ldots, t_n$ with respect to $l$, and if $P^\mathcal{M}(a_1, a_2, \ldots, a_n) = T$;
- in case $\phi$ is of the form $P$, if $P^\mathcal{M} = T$;
- in case $\phi$ has the form $\forall x \psi$, if the $\mathcal{M} \models_l [x \mapsto a] \psi$ holds for all $a \in A$;
- in case $\phi$ has the form $\exists x \psi$, if the $\mathcal{M} \models_l [x \mapsto a] \psi$ holds for some $a \in A$;
Satisfaction Relation (continued)

- in case $\phi$ has the form $\neg \psi$, if $M \models \neg \psi$ does not hold;
- in case $\phi$ has the form $\psi_1 \lor \psi_2$, if $M \models \psi_1$ holds or $M \models \psi_2$ holds;
- in case $\phi$ has the form $\psi_1 \land \psi_2$, if $M \models \psi_1$ holds and $M \models \psi_2$ holds; and
- in case $\phi$ has the form $\psi_1 \rightarrow \psi_2$, if $M \models \psi_2$ holds whenever $M \models \psi_1$ holds.
If a formula $\phi$ has no free variables, we call $\phi$ a *sentence*. $\mathcal{M} \models I \phi$ holds or does not hold regardless of the choice of $I$. Thus we write $\mathcal{M} \models \phi$ or $\mathcal{M} \not\models \phi$. 

The Importance of Being Formal

06—From Propositional to Predicate Logic
Semantic Entailment and Satisfiability

Let $\Gamma$ be a possibly infinite set of formulas in predicate logic and $\psi$ a formula.

**Entailment**

$\Gamma \models \psi$ iff for all models $M$ and environments $I$, whenever $M \models_I \phi$ holds for all $\phi \in \Gamma$, then $M \models_I \psi$.

**Satisfiability of Formulas**

$\psi$ is satisfiable iff there is some model $M$ and some environment $I$ such that $M \models_I \psi$ holds.

**Satisfiability of Formula Sets**

$\Gamma$ is satisfiable iff there is some model $M$ and some environment $I$ such that $M \models_I \phi$, for all $\phi \in \Gamma$. 
Let $\Gamma$ be a possibly infinite set of formulas in predicate logic and $\psi$ a formula.

**Validity**

$\psi$ is valid iff for all models $\mathcal{M}$ and environments $I$, we have $\mathcal{M} \models_I \psi$. 
The Problem with Predicate Logic

Entailment ranges over models
Semantic entailment between sentences: $\phi_1, \phi_2, \ldots, \phi_n \models \psi$
requires that in all models that satisfy $\phi_1, \phi_2, \ldots, \phi_n$, the sentence $\psi$ is satisfied.

How to effectively argue about all possible models?
Usually the number of models is infinite; it is very hard to argue on the semantic level in predicate logic.

Idea from propositional logic
Can we use natural deduction for showing entailment?
Relationship between propositional and predicate logic

If we consider propositions as nullary predicates, propositional logic is a sub-language of predicate logic.

Inheriting natural deduction

We can translate the rules for natural deduction in propositional logic directly to predicate logic.

Example

\[
\phi , \psi \quad \frac{}{\phi \land \psi} \quad [\land i] 
\]
Built-in Rules for Equality

\[ t = t \quad [x \Rightarrow t_1] \phi \]

\[ t_1 = t_2 \quad [x \Rightarrow t_2] \phi \]

\[ [= i] \quad [= e] \]
Properties of Equality

We show:

\[ f(x) = g(x) \vdash h(g(x)) = h(f(x)) \]

using

\[
\begin{align*}
  t_1 &= t_2 \\
  [x \Rightarrow t_1] \phi &\vdash [x \Rightarrow t_2] \phi \\\n  t = t &\vdash \phi \quad (= e)
\end{align*}
\]

1. \( f(x) = g(x) \) \quad premise
2. \( h(f(x)) = h(f(x)) \) \quad = i
3. \( h(g(x)) = h(f(x)) \) \quad = e \ 1,2
Elimination of Universal Quantification

Once you have proven $\forall x \phi$, you can replace $x$ by any term $t$ in $\phi$, provided that $t$ is free for $x$ in $\phi$. 

$$\forall x \phi$$

$$\frac{}{\forall x \ e} [\forall x \ e]$$

$$[x \Rightarrow t] \phi$$
Example

\[
\forall x \phi \\
\frac{\forall x \phi}{[\forall x \ t]} \\
[x \Rightarrow t] \phi
\]

We prove: \( S(g(john)) \), \( \forall x (S(x) \rightarrow \neg L(x)) \vdash \neg L(g(john)) \)

1. \( S(g(john)) \)  
   premise
2. \( \forall x (S(x) \rightarrow \neg L(x)) \)  
   premise
3. \( S(g(john)) \rightarrow \neg L(g(john)) \)  
   \( \forall x \ e \ 2 \)
4. \( \neg L(g(john)) \)  
   \( \rightarrow e \ 3,1 \)
Introduction of Universal Quantification

If we manage to establish a formula $\phi$ about a fresh variable $x_0$, we can assume $\forall x \phi$.

The variable $x_0$ must be fresh; we cannot introduce the same variable twice in nested boxes.
Example

\[ \forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \forall xQ(x) \text{ via } \forall x \phi \]

1. \( \forall x(P(x) \rightarrow Q(x)) \)  premise
2. \( \forall xP(x) \)  premise
3. \( P(x_0) \rightarrow Q(x_0) \)  \( \forall x \ e \ 1 \)  \( x_0 \)
4. \( P(x_0) \)  \( \forall x \ e \ 2 \)
5. \( Q(x_0) \)  \( \rightarrow e \ 3,4 \)
6. \( \forall xQ(x) \)  \( \forall x \ i \ 3-5 \)
In order to prove $\exists x \phi$, it suffices to find a term $t$ as “witness”, provided that $t$ is free for $x$ in $\phi$. 
Example

$$\forall x \phi \vdash \exists x \phi$$

Recall: Definition of Models

A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A non-empty set $U$, the universe;
2. ...

Remark

Compare this with Traditional Logic.

Because $U$ must not be empty, we should be able to prove the sequent above.
Example (continued)

\[ \forall x \phi \vdash \exists x \phi \]

1. \( \forall x \phi \)  
   \text{premise}
2. \([x \Rightarrow x] \phi\)  
   \(\forall x \ e \ 1\)
3. \(\exists x \phi\)  
   \(\exists x \ i \ 2\)
Elimination of Existential Quantification

\[ \exists x \varphi \]

Making use of \( \exists \)

If we know \( \exists x \varphi \), we know that there exist at least one object \( x \) for which \( \varphi \) holds. We call that element \( x_0 \), and assume \( [x \Rightarrow x_0] \varphi \). Without assumptions on \( x_0 \), we prove \( \chi \).
Example

\[ \forall x(P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x) \]

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \forall x(P(x) \rightarrow Q(x)) )</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>( \exists x P(x) )</td>
<td>premise</td>
</tr>
<tr>
<td>3</td>
<td>( P(x_0) )</td>
<td>assumption</td>
</tr>
<tr>
<td>4</td>
<td>( P(x_0) \rightarrow Q(x_0) )</td>
<td>( \forall x \ e \ 1 )</td>
</tr>
<tr>
<td>5</td>
<td>( Q(x_0) )</td>
<td>( \rightarrow \ e \ 4,3 )</td>
</tr>
<tr>
<td>6</td>
<td>( \exists x Q(x) )</td>
<td>( \exists x \ i \ 5 )</td>
</tr>
<tr>
<td>7</td>
<td>( \exists x Q(x) )</td>
<td>( \exists x \ e \ 2,3–6 )</td>
</tr>
</tbody>
</table>

Note that \( \exists x Q(x) \) within the box does not contain \( x_0 \), and therefore can be “exported” from the box.
### Another Example

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\forall x (Q(x) \rightarrow R(x))$</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>$\exists x (P(x) \land Q(x))$</td>
<td>premise</td>
</tr>
<tr>
<td>3</td>
<td>$P(x_0) \land Q(x_0)$</td>
<td>assumption, $x_0$</td>
</tr>
<tr>
<td>4</td>
<td>$Q(x_0) \rightarrow R(x_0)$</td>
<td>$\forall x \ e \ 1$</td>
</tr>
<tr>
<td>5</td>
<td>$Q(x_0)$</td>
<td>$\land e_2 \ 3$</td>
</tr>
<tr>
<td>6</td>
<td>$R(x_0)$</td>
<td>$\rightarrow e \ 4,5$</td>
</tr>
<tr>
<td>7</td>
<td>$P(x_0)$</td>
<td>$\land e_1 \ 3$</td>
</tr>
<tr>
<td>8</td>
<td>$P(x_0) \land R(x_0)$</td>
<td>$\land i \ 7, 6$</td>
</tr>
<tr>
<td>9</td>
<td>$\exists x (P(x) \land R(x))$</td>
<td>$\exists x \ i \ 8$</td>
</tr>
<tr>
<td>10</td>
<td>$\exists x (P(x) \land R(x))$</td>
<td>$\exists x \ e \ 2,3–9$</td>
</tr>
</tbody>
</table>
1. $\exists x P(x)$ \hspace{1cm} premise
2. $\forall x (P(x) \rightarrow Q(x))$ \hspace{1cm} premise
3. $P(x_0)$ \hspace{1cm} assumption $x_0$
4. $P(x_0) \rightarrow Q(x_0)$ \hspace{1cm} $\forall x \ e \ 2$
5. $Q(x_0)$ \hspace{1cm} $\rightarrow \ e \ 5,4$
6. $Q(x_0)$ \hspace{1cm} $\exists x \ e \ 1,4-6$
7. $\forall y Q(y)$ \hspace{1cm} $\forall y \ i \ 3-7$
Syntax of Predicate Logic

Predicate Logic as a Formal Language

Semantics of Predicate Logic

Proof Theory

Equivalences

Quantifier Equivalences

Soundness and Completeness
Equivalences

Two-way-provable

We write \( \phi \vdash \psi \) iff \( \phi \vdash \psi \) and also \( \psi \vdash \phi \).

Some simple equivalences

\[\begin{align*}
\neg \forall x \phi & \vdash \exists x \neg \phi \\
\neg \exists x \phi & \vdash \forall x \neg \phi \\
\forall x \forall y \phi & \vdash \forall y \forall x \phi \\
\exists x \exists y \phi & \vdash \exists y \exists x \phi \\
\forall x \phi \land \forall x \psi & \vdash \forall x (\phi \land \psi) \\
\exists x \phi \lor \exists x \psi & \vdash \exists x (\phi \lor \psi)
\end{align*}\]
\[ \neg \forall x \phi \vdash \exists x \neg \phi \]

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<tr>
<td>1</td>
<td>¬∀x(\phi)</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>¬∃x(\neg \phi)</td>
<td>assumption</td>
</tr>
<tr>
<td>3</td>
<td>(x_0)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(\neg [x \Rightarrow x_0] \phi)</td>
<td>assumption</td>
</tr>
<tr>
<td>5</td>
<td>(\exists x \neg \phi)</td>
<td>(\exists x\ i\ 4)</td>
</tr>
<tr>
<td>6</td>
<td>(\bot)</td>
<td>(\neg e\ 5, 2)</td>
</tr>
<tr>
<td>7</td>
<td>([x \Rightarrow x_0] \phi)</td>
<td>PBC 4–6</td>
</tr>
<tr>
<td>8</td>
<td>(\forall x \phi)</td>
<td>(\forall x\ i\ 3–7)</td>
</tr>
<tr>
<td>9</td>
<td>(\bot)</td>
<td>(\neg e\ 8, 1)</td>
</tr>
<tr>
<td>10</td>
<td>(\exists x \neg \phi)</td>
<td>PBC 2–9</td>
</tr>
</tbody>
</table>
Assume that $x$ and $y$ are different variables.

1. $\exists x \exists y \phi$  \hspace{1cm} \text{premise}

2. $[x \Rightarrow x_0](\exists y \phi)$  \hspace{1cm} \text{assumption}

3. $\exists y([x \Rightarrow x_0] \phi)$  \hspace{1cm} \text{def of subst ($x$, $y$ different)}

4. $[y \Rightarrow y_0][x \Rightarrow x_0] \phi$  \hspace{1cm} \text{assumption}

5. $[x \Rightarrow x_0][y \Rightarrow y_0] \phi$  \hspace{1cm} \text{def of subst ($x$, $y$, $x_0$, $y_0$ different)}

6. $\exists x[y \rightarrow y_0] \phi$  \hspace{1cm} $\exists x \ i \ 5$

7. $\exists y \exists x \phi$  \hspace{1cm} $\exists y \ i \ 6$

8. $\exists y \exists x \phi$  \hspace{1cm} $\exists y \ e \ 3, 4–7$

9. $\exists y \exists x \phi$  \hspace{1cm} $\exists x \ e \ 1, 2–8$
Assume that $x$ is not free in $\psi$

\[
\begin{align*}
\forall x \phi \land \psi & \iff \forall x (\phi \land \psi) \\
\forall x \phi \lor \psi & \iff \forall x (\phi \lor \psi) \\
\exists x \phi \land \psi & \iff \exists x (\phi \land \psi) \\
\exists x \phi \lor \psi & \iff \exists x (\phi \lor \psi)
\end{align*}
\]
Central Result of Natural Deduction

\[ \phi_1, \ldots, \phi_n \models \psi \]

iff

\[ \phi_1, \ldots, \phi_n \vdash \psi \]

proven by Kurt Gödel, in 1929 in his doctoral dissertation (just one year before his most famous result, the incompleteness results of mathematical logic)