Graph Cycles and Olympiad Problems

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We present four simple lemmas concerning the existence of cycles in graphs, and show how the lemmas can be applied to solve problems that appeared in mathematical competitions. Despite their simplicity, these tools can be used to tackle surprisingly varied problems.

We begin with some basics of graph theory. A *graph* G consists of a set of *vertices* V and a set of *edges* E. The graph is said to be *finite* if both V and E are finite.

In a *directed graph*, each edge is directed from a vertex v to a vertex w, where possibly $v = w$. We allow multiple edges to be directed between the same pair of vertices. A *directed cycle* is a sequence of vertices $v_1, v_2, \ldots, v_n$, for some $n \geq 1$, such that there is a directed edge $v_i \to v_{i+1}$ for all $i = 1, 2, \ldots, n-1$ and a directed edge $v_n \to v_1$. (If $n = 1$, we require a directed edge $v_1 \to v_1$.) The *out-degree* of a vertex is the number of edges directed from that vertex, and the *in-degree* of a vertex is the number of edges directed into that vertex.

Similarly, in an *undirected graph*, each edge connects two vertices v and w, where possibly $v = w$. We allow multiple edges to connect the same pair of vertices. An *undirected cycle* is a sequence of vertices $v_1, v_2, \ldots, v_n$, for some $n \geq 1$, such that there is an undirected edge between v_i and v_{i+1} for all $i = 1, 2, \ldots, n-1$ and an undirected edge between v_n and v_1. (If $n = 1$, we require an undirected edge between v_1 and itself.) The *degree* of a vertex is the number of edges adjacent to that vertex. (An edge between a vertex and itself is counted twice in the degree of that vertex.)

We are now ready to state the lemmas. First, we consider directed graphs.

Lemma 1. *Let G be a finite directed graph. If every vertex of G has out-degree at least* 1*, then G has a directed cycle.*

Proof. Suppose that every vertex of G has out-degree at least 1. Let v_1 be an arbitrary vertex of G. Given vertex v_i, define v_{i+1} as any vertex for which there exists an edge $v_i \to v_{i+1}$. Since the graph is finite, there exist two indices $j < k$ such that $v_j = v_k$. Then the path $v_j \to v_{j+1} \to \cdots \to v_k$ forms a directed cycle. ■

Lemma 2. *Let G be a finite directed graph. If every vertex of G has out-degree* 1 *and in-degree* 1*, then G is a disjoint union of directed cycles.*

Proof. Suppose that every vertex of G has out-degree 1 and in-degree 1. Let v_1 be an arbitrary vertex of G. Given vertex v_i, define v_{i+1} as the vertex for which there exists an edge $v_i \to v_{i+1}$. Since the graph is finite, there exist two indices $j < k$ such that $v_j = v_k$. Consider the first vertex v_k such that there exists some $j < k$ for which $v_j = v_k$. If $j > 1$, then v_j has in-degree at least 2, contradicting the assumption. Hence $j = 1$, and the path $v_1 \to v_2 \to \cdots \to v_{k-1} \to v_1$ forms a directed cycle. We can remove it and apply mathematical induction on the remaining graph, which consists of strictly fewer vertices. ■

We also have equivalent results on undirected graphs. The proofs are similar and are left as exercises for the reader.

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Lemma 3. *Let G be a finite undirected graph. If every vertex of G has degree at least 2, then G has an undirected cycle.*

Lemma 4. *Let G be a finite undirected graph. If every vertex of G has degree 2, then G is a disjoint union of undirected cycles.*

The key in applying these lemmas is to identify the appropriate graph G for each problem. We illustrate this through some problems given in mathematical competitions.

Our first problem involves a simple fact from matching theory, an extremely rich subject on its own (see Roth and Sotomayor [**1**], for example). In the language of matching theory, it states that in a "one-sided matching," for every assignment in the "core" of the game, at least one person receives his or her top choice. The problem appeared in the Turkish Mathematical Olympiad in 1998, and is closely related to the Top Trading Cycle Algorithm [**2**].

Problem 1. *There are n people who need to be assigned to n houses. Each person ranks the houses in some order, with no ties. After the assignment is made, it is observed that every other assignment would assign at least one person to a house that the person ranks lower than the house in the given assignment. Prove that at least one person receives his or her top choice in the given assignment.*

Proof. Assume that the ith person has been assigned to the ith house. Construct a directed graph G with vertices $1, 2, \dots, n$. For each person i, add an edge from i to i's favorite house. Every vertex of G has out-degree 1, so by Lemma 1, G contains a directed cycle. If the cycle consists of a single vertex, we have found a person who receives his or her top choice in the given assignment. Otherwise, we can let the people in the cycle trade their houses along the cycle, making all of them happier and thus contradicting the condition of the problem. ■

Our next problem appeared in the Iranian Mathematical Olympiad in 1998.

Problem 2. *An $n \times n$ table is filled with the numbers $-1, 0, 1$ in such a way that every row and column of the table contains exactly one -1 and one 1. (An example is shown below.) Prove that one can permute the rows and columns so that in the resulting table each number is the negative of the number in the same position in the original table.*

Proof. Consider a directed graph G whose vertices correspond to the rows of the table. For each column, add an edge to G pointing from the row in which the column has a 1 to the row in which the column has a -1. For example, for this table:

$$\begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

the graph G consists of the edges $4 \to 2 \to 1 \to 3 \to 4$. Every vertex of G has out-degree 1 and in-degree 1, so by Lemma 2, G is a disjoint union of directed cycles. Consider any directed cycle

$$s_1 \to s_2 \to \cdots \to s_k \to s_1$$

in G, and assume that row s_i has a 1 in column t_i. (For the cycle $4 \to 2 \to 1 \to 3 \to 4$ in the above table, we have $t_1 = 1$, $t_2 = 2$, $t_3 = 4$, and $t_4 = 3$.) We switch rows s_2

and s_k, rows s_3 and $s_{k-1}, \ldots,$ and rows $s_{\lceil k/2 \rceil}$ and $s_{\lfloor k/2 \rfloor+2}$. Note that this has the effect of reversing the cycle. We then switch columns t_1 and t_k, columns t_2 and $t_{k-1}, \ldots,$ and columns $t_{\lfloor k/2 \rfloor}$ and $t_{\lceil k/2 \rceil+1}$. It can be verified that after this procedure, all 1's in these rows and columns are replaced by -1's, and vice versa. Hence, after we perform the procedure on all cycles in G, we obtain the desired table. ■

Our next problem appeared in the Russian Mathematical Olympiad in 2005.

Problem 3. *A conference has* 100 *participants from* 50 *countries, two from each country. The participants sit at a round table. Prove that one may partition them into two groups in such a way that no two participants from the same country are in the same group, and no three consecutive participants in the circle are in the same group.*

Proof. Number the participants in the round table $1, 2, \ldots, 100$ in the order in which they are seated, and pair them up as $\{1, 2\}, \{3, 4\}, \ldots, \{99, 100\}$. Construct an undirected graph G with the vertices corresponding to the participants. For each participant, add an edge to her pair and an edge to the other participant from the same country. Every vertex of G has degree 2, so by Lemma 4, G is a disjoint union of undirected cycles. Since the edges in a cycle necessarily alternate between the "pair" type and the "same country" type, each cycle must be of even length. Hence, we can partition the participants in each cycle into two groups so that no two participants in the same group are connected by an edge. It follows that no two participants from the same country are in the same group, and no three consecutive participants in the circle are in the same group. ■

Our next problem appeared in the Peru Team Selection Test for the International Mathematical Olympiad in 2006.

Problem 4. *A table with* 2^n *rows and* n *columns is filled with* 1 *and* -1 *in such a way that the rows of the table constitute all possible sequences of length* n *that can be formed with* 1 *and* -1. *An arbitrary subset of numbers is then replaced by* 0*s. (An example is shown below.) Prove that one can choose a nonempty subset of rows of the table so that within the chosen rows, the sum of the numbers in each column is 0.*

$$\begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Proof. We refer to a row in the table before modification as an *original row*, and one after modification as a *modified row*. We define a *binary row* to be a row with n entries consisting of 0 and 1. Let

$$f : \{0, 1\}^n \to \{-1, 1\}^n$$

be a function such that for each binary row b, $f(b)$ is the original row for which the entry of $f(b)$ is 1 if the corresponding entry of b is 0, and the entry of $f(b)$ is -1 if the corresponding entry of b is 1. (For example, $f(0, 0) = (1, 1)$ and $f(1, 0) = (-1, 1)$.) Let

$$g : \{0, 1\}^n \to \{-1, 0, 1\}^n$$

be such that $g(b)$ is the modified row that coincides with $f(b)$ before we made any change to the table, but possibly has had some entries replaced by 0. (So in the above table, $g(0, 0) = (1, 1)$ and $g(1, 0) = (-1, 0)$.) One can check that $b + g(b)$ is again a binary row, where addition is done entrywise. Construct a directed graph G with the vertices corresponding to all 2^n binary rows. For each binary row b, add an edge to the binary row $b + g(b)$. Every vertex of G has out-degree 1, so by Lemma 1, G contains a directed cycle $b_1, b_2, \ldots, b_k$. Hence

$$b_1 + g(b_1) + g(b_2) + \cdots + g(b_k) = b_1,$$

which implies that $g(b_1), g(b_2), \ldots, g(b_k)$ form a desired subset of rows. ■

Our final problem appeared on the shortlist of the International Mathematical Olympiad in 2017.

Problem 5. *Determine all integers $n \geq 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers*

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

Proof. First, if n is composite, say $n = ab$ for integers $a, b \geq 2$, then we can take

$$(a_1, a_2, \ldots, a_{n-1}, a_n) = (a, a, \ldots, a, 0)$$

to show that the property does not hold.

Next, let n be a prime number, and assume for the sake of contradiction that the property does not hold. Construct a directed graph G with vertices $1, 2, \ldots, n$. For each a_i, there exists $1 \leq j \leq n - 1$ such that

$$a_i + a_{i+1} + \cdots + a_{i+j-1}$$

is divisible by n. Add a directed edge from i to $i + j$ in G, where we take the vertex indices modulo n. Every vertex of G has out-degree 1, so by Lemma 1, G contains a directed cycle $v_1, v_2, \ldots, v_k$. This means that the sum

$$(a_{v_1} + a_{v_1+1} + \cdots + a_{v_2-1}) + (a_{v_2}+a_{v_2+1} + \cdots + a_{v_3-1}) + \cdots + (a_{v_k} + a_{v_k+1} + \cdots + a_{v_1-1})$$

is divisible by n. This sum contains each a_i an equal number of times, and this number can be at most $n - 1$. Hence, we have that $r(a_1 + a_2 + \cdots + a_n)$ is divisible by n for some $1 \leq r \leq n - 1$. However, this is impossible since n is prime and neither r nor $a_1 + a_2 + \cdots + a_n$ is divisible by n. ■

REFERENCES

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[2] Shapley, L., Scarf, H. (1974). On cores and indivisibility. *J. Math. Econ.* 1(1): 23–37. doi.org/10.1016/0304-4068(74)90033-0

Summary. We show how certain basic results about cycles in directed and undirected graphs can be used to solve some clever problems that appeared in major mathematics competitions.

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