Fair Division of Indivisible Items: Asymptotics and Graph-Theoretic Approaches

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Resource Allocation

Resource allocation problems are everywhere! For example, we commonly need to allocate

- school supplies to children
- course slots in universities to students
- machine processing time to users
- kidneys to kidney transplant patients
- etc.

"Understanding who gets what, and how and why, is still very much a work in progress."

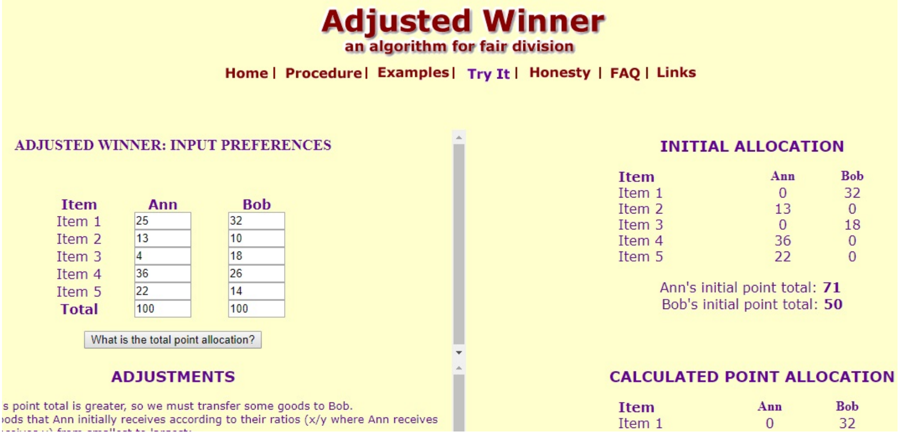
— Alvin E. Roth, 2012 Nobel Laureate in Economics

Fair Division

- The history of fair division dates back to the Bible.

> – There were quarrels between the herders of Abram's livestock and the herders of Lot's livestock.
> – So Abram said to Lot: "Let there be no strife between you and me . . . Is not the whole land available? Please separate from me. If you prefer the left, I will go to the right; if you prefer the right, I will go to the left."
> – Lot looked about and saw how abundantly watered the whole Jordan Plain was.
> – Lot, therefore, chose for himself the whole Jordan Plain . . . Abram settled in the land of Canaan, while Lot settled among the cities of the Plain.
>
> — Book of Genesis, Chapter 13

- This is an example of cut and choose, a classical fair division protocol.



www.nyu.edu/projects/adjustedwinner

Fair Division

Divide Your Rent Fairly

APRIL 28, 2014

When you're sharing an apartment with roommates, it can be a challenge to decide who takes which bedroom, and at what price. Sit down with your roommates and use the calculator below to find the fair division. **RELATED ARTICLE**

What's your total rent? $ 1000 **How many of you are there?** 2 **3** 4 5 6 7 8

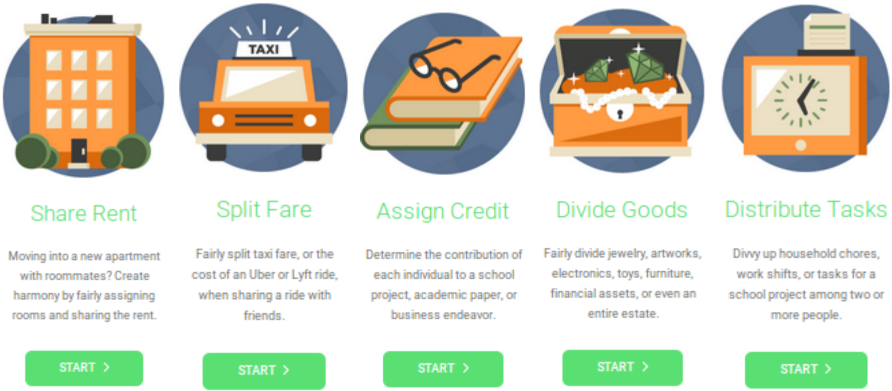
If the rooms have the following prices, which room would you choose?

Choices will not necessarily be in order and the same roommate may be asked to choose multiple times in a row. Each roommate keeps choosing until a fair division is found.

Roommate A	☐ **$63** Room 1	☐ **$813** Room 2	☐ **$125** Room 3
Roommate B	$63 Room 1	$844 Room 2	**$94 Room 3**
Roommate C	**$31 Room 1**	$875 Room 2	$94 Room 3

www.nytimes.com/interactive/2014/science/rent-division-calculator.html

Fair Division



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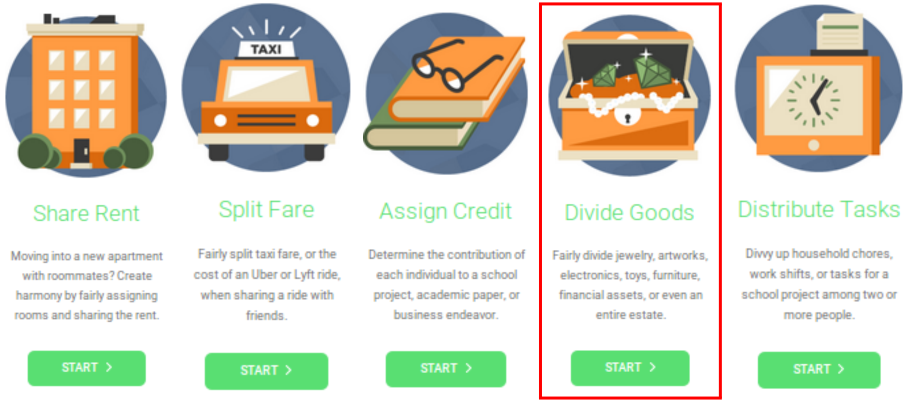
Fair Division



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Fairness Notions

There are n agents. Every agent i has a valuation function u_i and receives a (possibly empty) part M_i of the resource M.

- Envy-freeness: Each agent values her part at least as much as any other agent's part — $u_i(M_i) \geq u_i(M_j)$ for all i, j.
- Proportionality: Each agent values her part at least $1/n$ of her value for the entire resource — $u_i(M_i) \geq \frac{1}{n} \cdot u_i(M)$ for all i.
- Equitability: All agents receive the same value for their own part — $u_i(M_i) = u_j(M_j)$ for all i, j.
- When the resource is divisible, all three properties can be satisfied.
- This is not true when the resource consists of indivisible items, for example if there are two agents and one valuable item.

Model

- Let $N = \{1, 2, \ldots, n\}$ be the agents and $M = \{1, 2, \ldots, m\}$ be the indivisible items (or goods).
- An allocation is an assignment of each item to at most one agent. It is said to be complete if all items are assigned.
- Each agent i has a value $u_i(M')$ for each set of items $M' \subseteq M$.
- It is usually assumed that the valuations are
  - Normalized: $u(\emptyset) = 0$.
  - Monotonic: $u(M_1) \leq u(M_2)$ for $M_1 \subseteq M_2 \subseteq M$.
- Sometimes it is further assumed that the valuations are
  - Additive: $u(M') = \sum_{j \in M'} u(j)$ for $M' \subseteq M$.
  - Binary additive (or binary): u is additive and $u(j) \in \{0, 1\}$ for every item $j \in M$.

Proportionality

- An allocation is proportional if $u_i(M_i) \geq \frac{1}{n} \cdot u_i(M)$ for all i.
- Assume that utilities are additive.
- The utilities $u_i(j)$ are drawn independently from a distribution $\mathcal{U}$ (e.g., the uniform distribution over $[0, 1]$).
- If $m < n$, no allocation is proportional.
- What if $m = n$?

Theorem [S., 2016]

If $m = n$, then with high probability, there exists a proportional allocation.

- An event occurring "with high probability" means that the probability that it occurs approaches 1 as $n \to \infty$.

Chernoff Bound

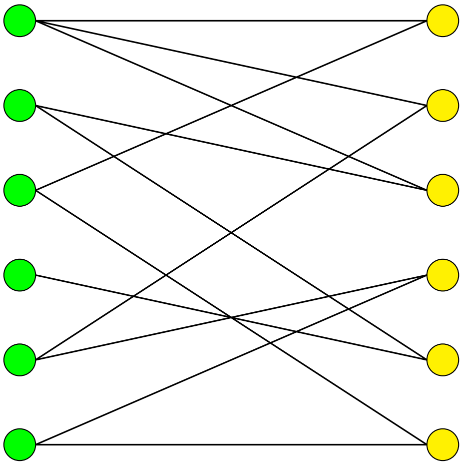
Theorem [Chernoff, 1952]

Let $X_1, X_2, \ldots, X_k$ be independent random variables in $[0, 1]$, and let $X = X_1 + X_2 + \cdots + X_k$. Then for any $\epsilon \in (0, 1)$:

$$\Pr[X \geq (1+\epsilon)E[X]] \leq e^{-\frac{\epsilon^2 E[X]}{3}}.$$

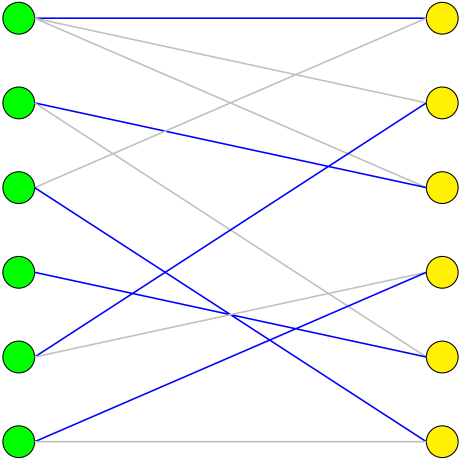
- A similar bound holds for the opposite inequality.
- If we flip a fair coin n times, the number of heads is likely to be close to $n/2$.
- In our allocation problem, for each agent, the proportional share $\frac{1}{n} \cdot u_i(M)$ is likely to be close to $1/2$.
- With high probability, it is sufficient to give every agent an item of value at least $2/3$.

Random Matching



Bipartite graph

Random Matching



Perfect matching

Random Matching

- Erdős-Rényi random graph model: Each edge in the bipartite graph is present with probability p, independently of other edges.

Theorem [Erdős/Rényi, 1964]

If $p = \log n/n + \omega(1/n)$, then with high probability, the graph contains a perfect matching.

- We add an edge between an agent and an item if the agent values the item at least $2/3$ (i.e., the item is valuable enough for the agent).
- The probability that each edge is present is $1/3$, so there is a perfect matching with high probability.
- The argument generalizes to $m = kn$ for constant k, and $m = \omega(n)$.

Envy-Freeness

- An allocation is envy-free if $u_i(M_i) \geq u_i(M_j)$ for all i, j.
- For additive utilities, envy-freeness implies proportionality.
  - **Proof:** Fix i. We have $n \cdot u_i(M_i) \geq u_i(M_1) + \cdots + u_i(M_n) = u_i(M)$, so $u_i(M_i) \geq u_i(M)/n$.
- If $m < n$, no envy-free allocation exists.
- If $m = n$, every agent must receive exactly one item. This means all agents must have different top items, which is very unlikely!

Theorem [Dickerson et al., 2014]

When $m = n + o(n)$, an envy-free allocation is still unlikely to exist.

- **Proof idea:** If k agents have the same top item, only one of them can get it. The remaining $k-1$ agents must receive at least two items each. There are not enough items to go around.

Envy-Freeness

Theorem [Dickerson et al., 2014]

When $m = \Omega(n \log n)$, an allocation that maximizes social welfare (i.e., sum of agents' utilities) is envy-free with high probability.

Proof sketch:

- Fix i, j. We claim that agent i envies agent j with probability $o(1/n^2)$.
- If this is true, a union bound over all i, j implies that with high probability, no agent envies another agent.
- Each time i receives an item, her expected value is $\geq 2/3$.
- Each time j receives an item, i's expected value is $\leq 1/2$.
- To use Chernoff bound, we need $e^{-\frac{\epsilon^2 E[X]}{3}}$ to be $o(1/n^2)$.
  So $E[X]$ must be $\Omega(\log n)$, implying that we need $m = \Omega(n \log n)$.

Envy-Freeness

- Let $m = n \log n - \omega(n)$. With high probability, the welfare-maximizing allocation is not envy-free. [Manurangsi/S., 2019]
  - Follows from the coupon collector's problem [Erdős/Rényi, 1961].
- Where is the transition between $m = n + o(n)$ and $m = \Omega(n \log n)$?
- Surprisingly, there is no universal point of transition!

Theorem [Manurangsi/S., 2019]

– If m is divisible by n, an EF allocation exists with high probability as long as $m \geq 2n$.
– Else, an EF allocation is unlikely even when $m = \Theta(n \log n / \log \log n)$.

EF1: Round-Robin Algorithm

- What if we want a fairness notion that can always be satisfied?
- Envy-freeness up to one good (EF1): Agent i may envy agent j, but the envy can be eliminated by removing an item from j's bundle.
- Can be satisfied by the round-robin algorithm: Let the agents take turns choosing their favorite item from the remaining items.
- If i is ahead of j in the round-robin ordering, then in every "round", i does not envy j.
- If i is behind j in the ordering, we consider the first round to start with i's first pick. Then i does not envy j up to j's first item.
- **Bonus:** The resulting allocation is always balanced.

EF1: Maximum Nash Welfare

- The Nash welfare of an allocation is the product of the agents' utilities: $\prod_{i=1}^{n} u_i(M_i)$.

Theorem [Caragiannis et al., 2016]

An allocation that maximizes the Nash welfare, called the maximum Nash welfare (MNW) allocation, is EF1.

- **Proof sketch:**
  - Suppose for contradiction that agent i envies agent j even after removing any item from j's bundle.
  - Consider an item k in j's bundle that minimizes the ratio $u_j(k)/u_i(k)$.
  - Moving k to i's bundle increases the Nash welfare.
- **Bonus:** The resulting allocation is always Pareto optimal: we cannot make some agent better off without making another agent worse off.

EF1: Envy-Cycle Elimination Algorithm

- What about arbitrary monotonic valuations?
- We can still obtain an EF1 allocation using the envy-cycle elimination algorithm [Lipton et al., 2004].
  1. Allocate one good at a time in arbitrary order.
  2. Maintain an envy graph with the agents as its vertices, and a directed edge $i \rightarrow j$ if i envies j with respect to the current (partial) allocation.
  3. At each step, the next good is allocated to an agent with no incoming edge. Any cycle that arises is eliminated by giving j's bundle to i for any edge $i \rightarrow j$ in the cycle.
- **Invariant:** The envy graph has no cycles and therefore has an agent with no incoming edge before each allocation of a good.
- The algorithm runs in time $O(n^3 m)$, even though the utility functions can have exponential size.

EFX

- Envy-freeness up to any good (EFX): Agent i may envy agent j, but the envy can be eliminated by removing any item from j's bundle.
- An EFX allocation always exists for identical monotonic valuations [Plaut/Roughgarden, 2018].
  - **Proof idea:** Use the leximin allocation, which maximizes the minimum utility, then the second minimum utility, and so on.
- Cut-and-choose implies existence for two agents with arbitrary monotonic valuations.
- The question remains open for three or more agents!
- For additive valuations, there is an EFX allocation of a subset of items with Nash welfare at least half of the MNW for the original set [Caragiannis et al., 2019].

Summary

- There is a rich theory of fair division for a variety of situations.
- Classical fairness notions like envy-freeness and proportionality cannot always be satisfied for indivisible items.
- For proportionality, existence is likely even for $m = n$.
- On the other hand, for envy-freeness we need $m \geq 2n$ if m is divisible by n, and $m = \Omega(n \log n)$ otherwise.
- These results often rely on Chernoff bounds and random graph theory.
- Envy-freeness up to one good (EF1) can always be satisfied. Different EF1 algorithms come with different extra properties.

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