Fair Division of Indivisible Items: Asymptotics and Graph-Theoretic Approaches

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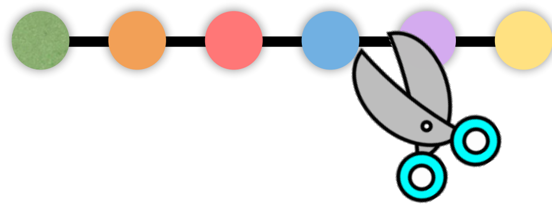
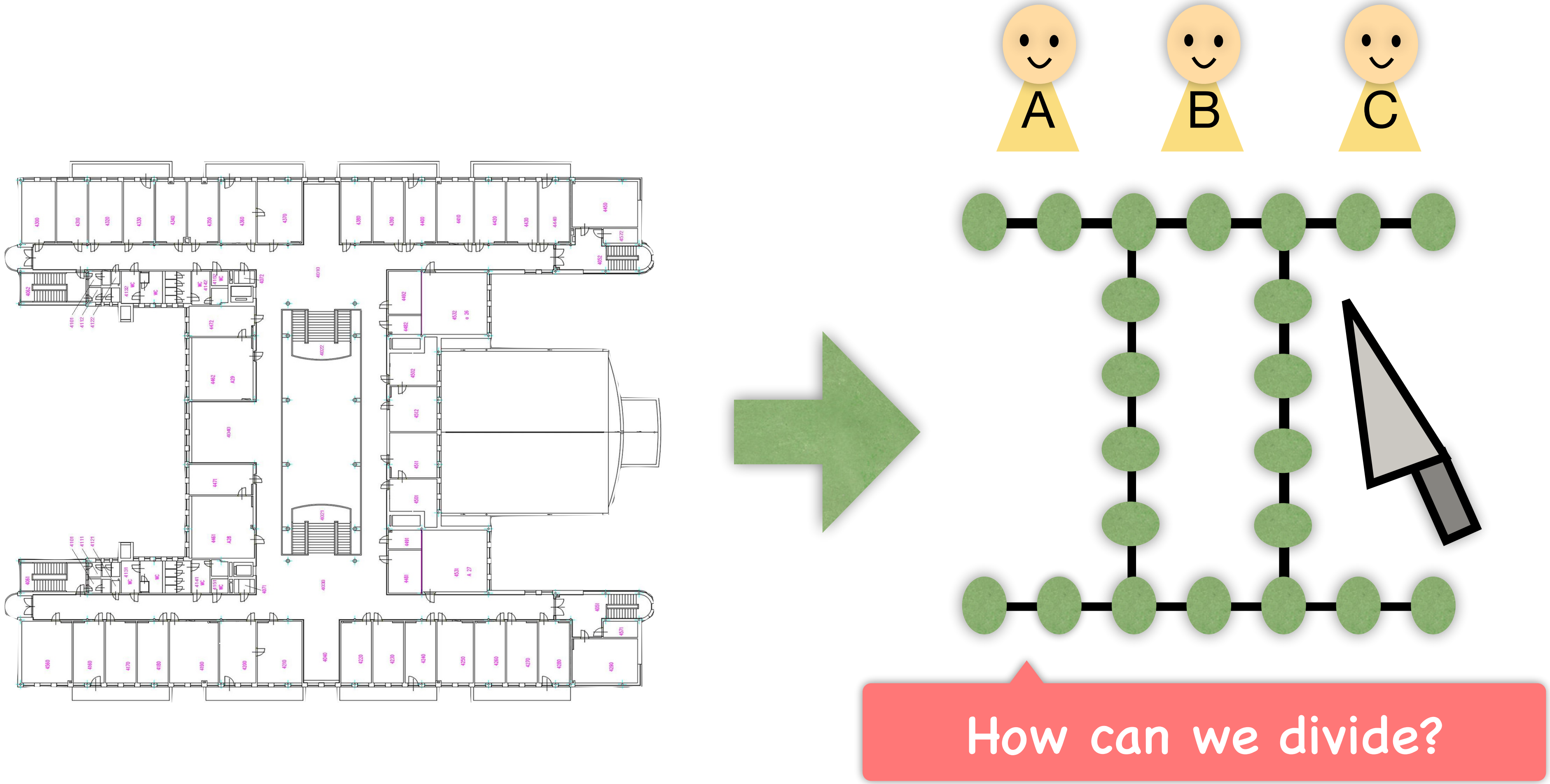
Some slides credit from Dominik Peters



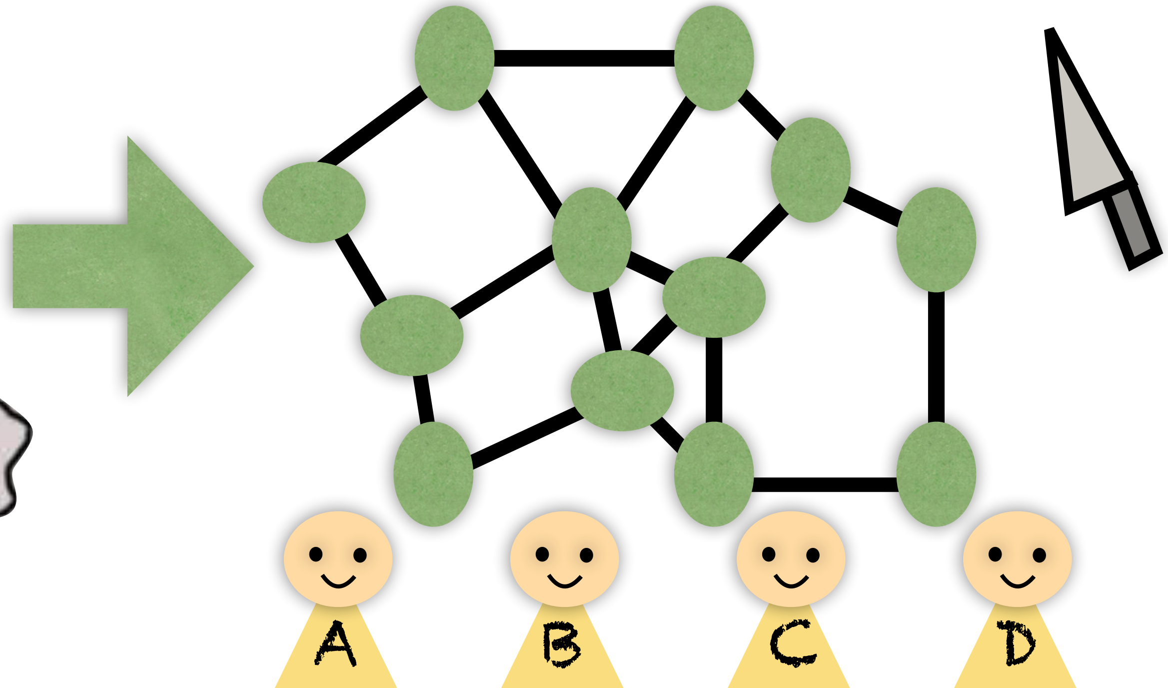
Fair division of a graph

- Office allocation: Allocate a connected set of rooms to each research group.



Fair division of a graph

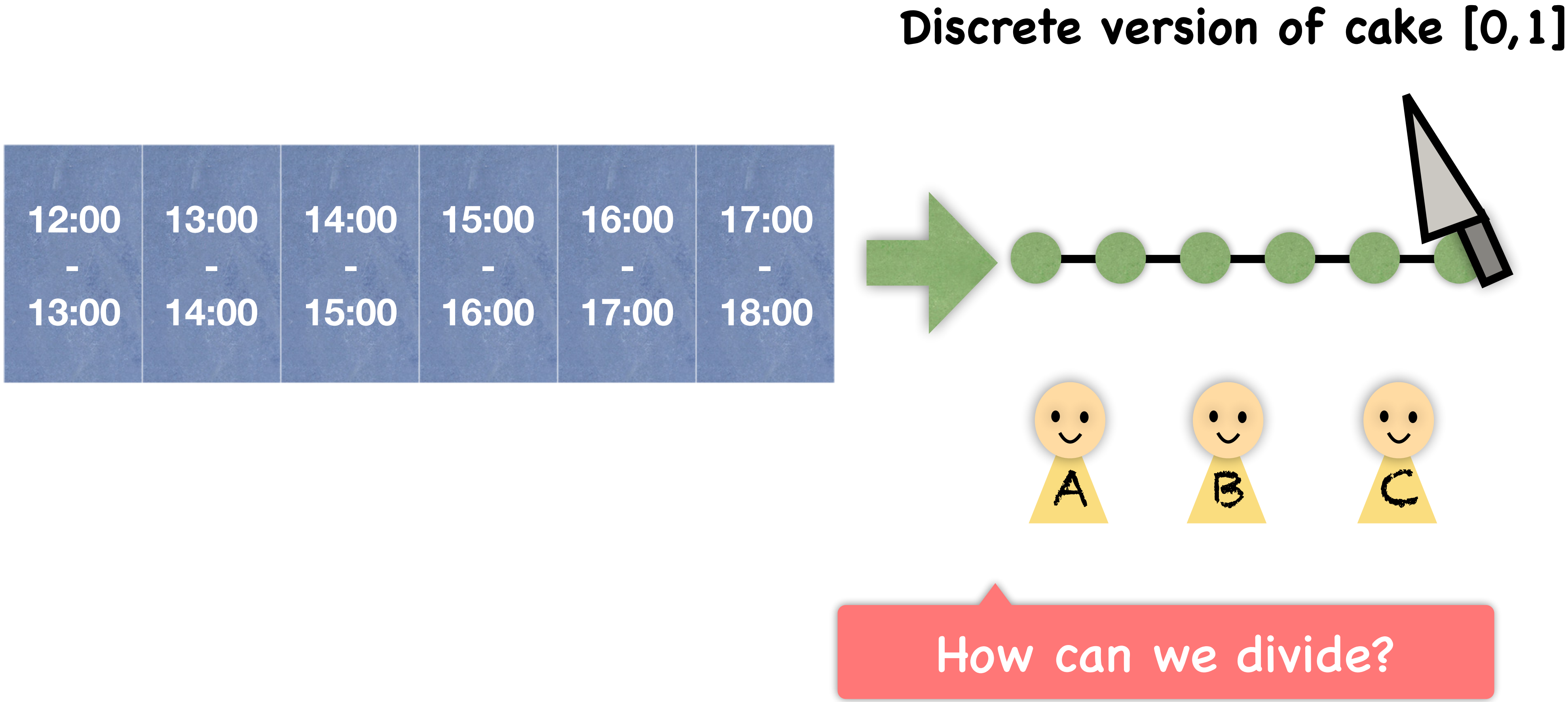
- Land division: Allocate a connected set of regions to each country.



How can we divide?

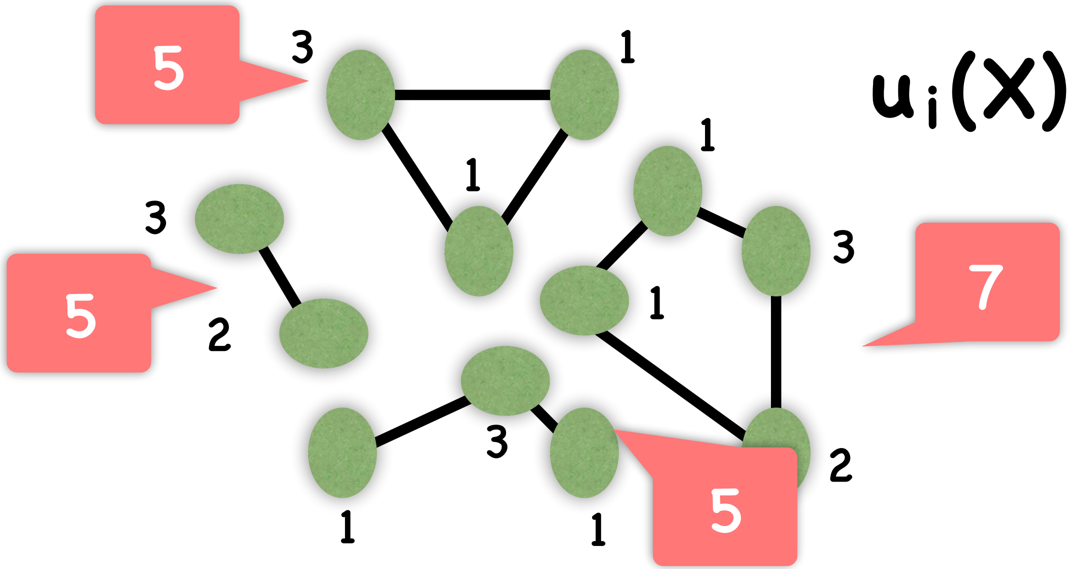
Fair division of a graph

- Scheduling: Allocate a connected set of time slots to each agent.



Model [Bouveret et al. 2017]

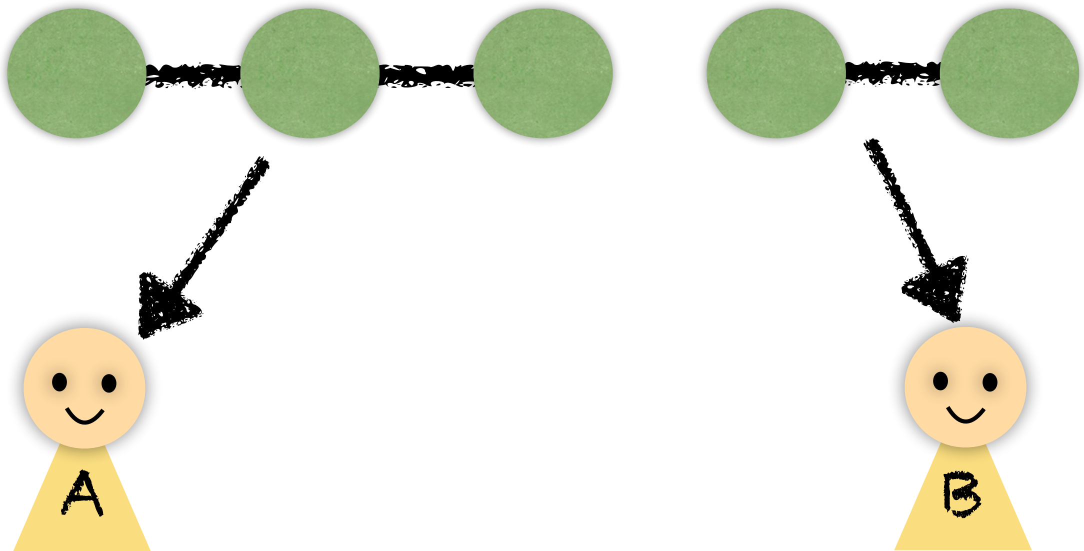
- An undirected graph $G=(V,E)$
- A set of agents $N = \{1,2,\ldots,n\}$
- A non-negative additive utility function $u_i : V \to R_+$



$$u_i(X) = \sum_{v \text{ in } X} u_i(v)$$

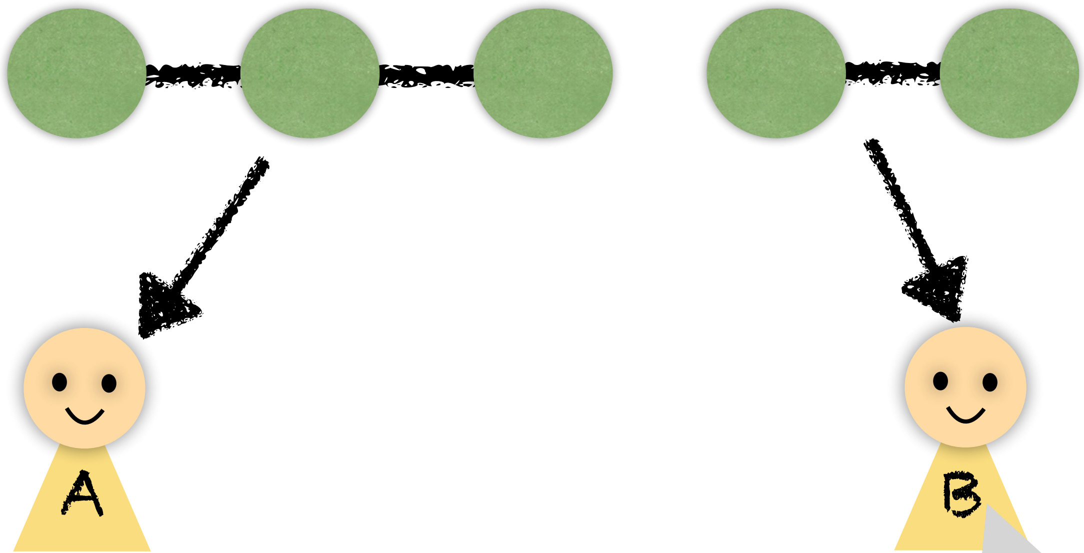
Model

- A connected allocation is a mapping assigning each player to a disjoint connected subset of the vertices.



Classical fairness notions

- A connected allocation is envy-free if no one envies others:
  u_i(i's bundle) $\geq$ u_j(j's bundle) for all i,j in N

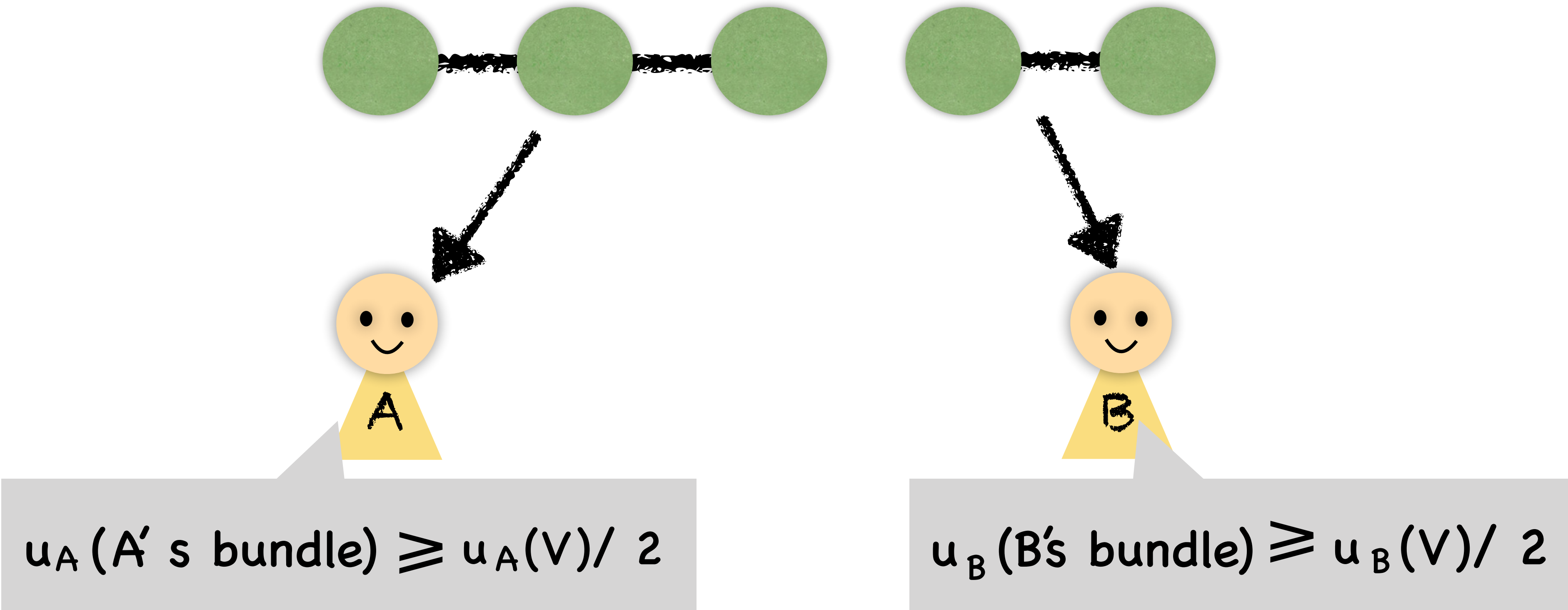


u_A(A's bundle) $\geqslant$ u_A(B's bundle)

u_B(B's bundle) $\geqslant$ u_B(A's bundle)

Classical fairness notions

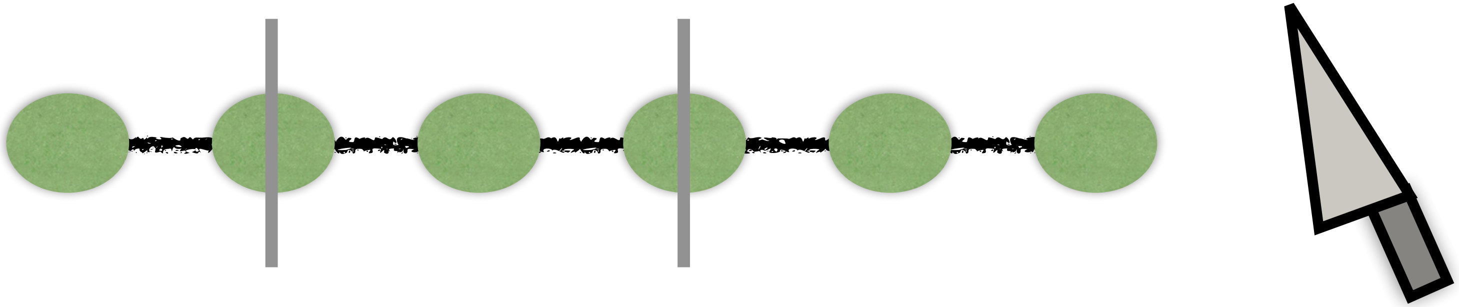
- A connected allocation is proportional if each player receives value ⩾ his proportional share: u_i(i's bundle) ⩾ $u_i(V)/n$ for all i in N



Existence of EF and Prop

- Proportional/envy-free contiguous allocation of a cake [0,1] exists with divisibilities.

	Existence	Complexity
Envy-freeness	✓ [Stromquist, 1980]	no finite protocol [Stromquist, 2008]
Proportionality	✓ [Dubins and Spanier, 1961]	polytime [Dubins and Spanier, 1961]



Approximate fairness

- Proportional/envy-free allocation may not exist with indivisibilities → Relaxations?

Consider an instance of two players and one item.

Approximate fairness

- Proportional/envy-free allocation may not exist with indivisibilities → Relaxations?
- Budish (2011) proposed the following two concepts:
  - Maximin share (MMS)
  - Envy-freeness up to one good (EF1)

Approximate fairness

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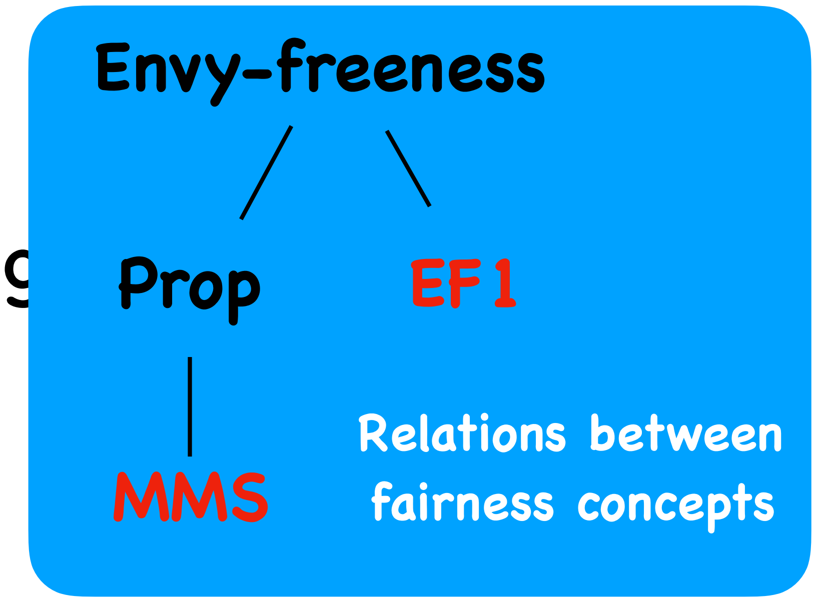
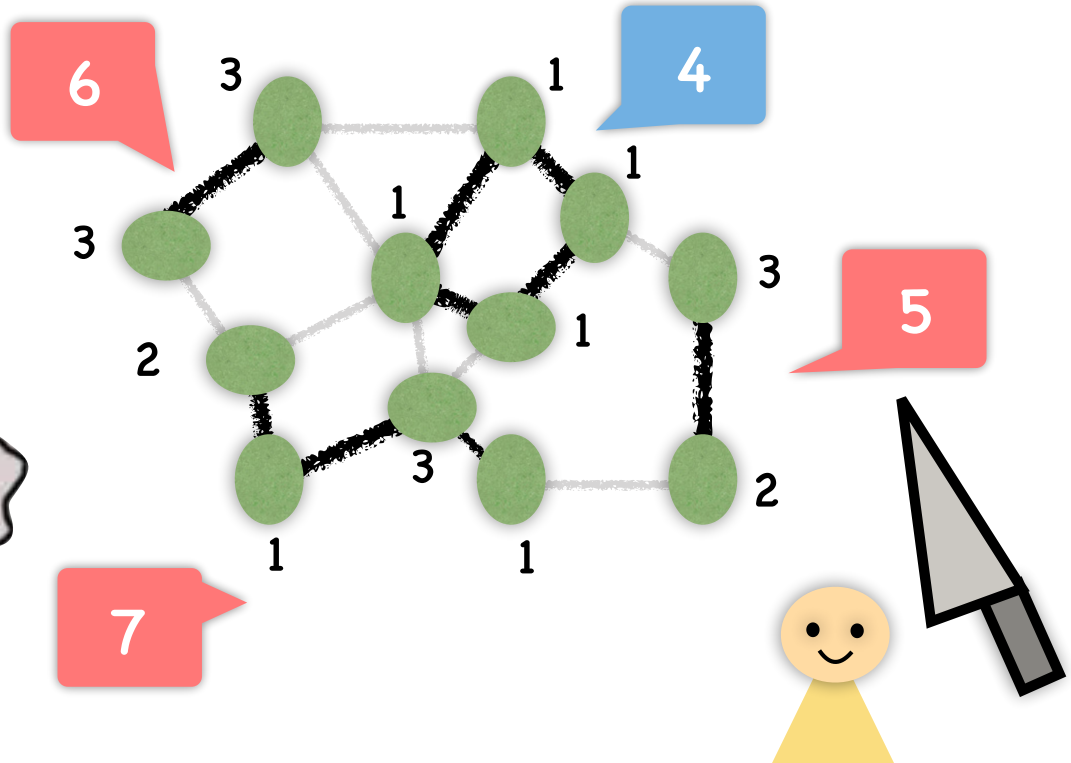


Maximin share

- Maximin share [Budish, 2011]: the best utility each agent would receive if she had to cut and choose the last.



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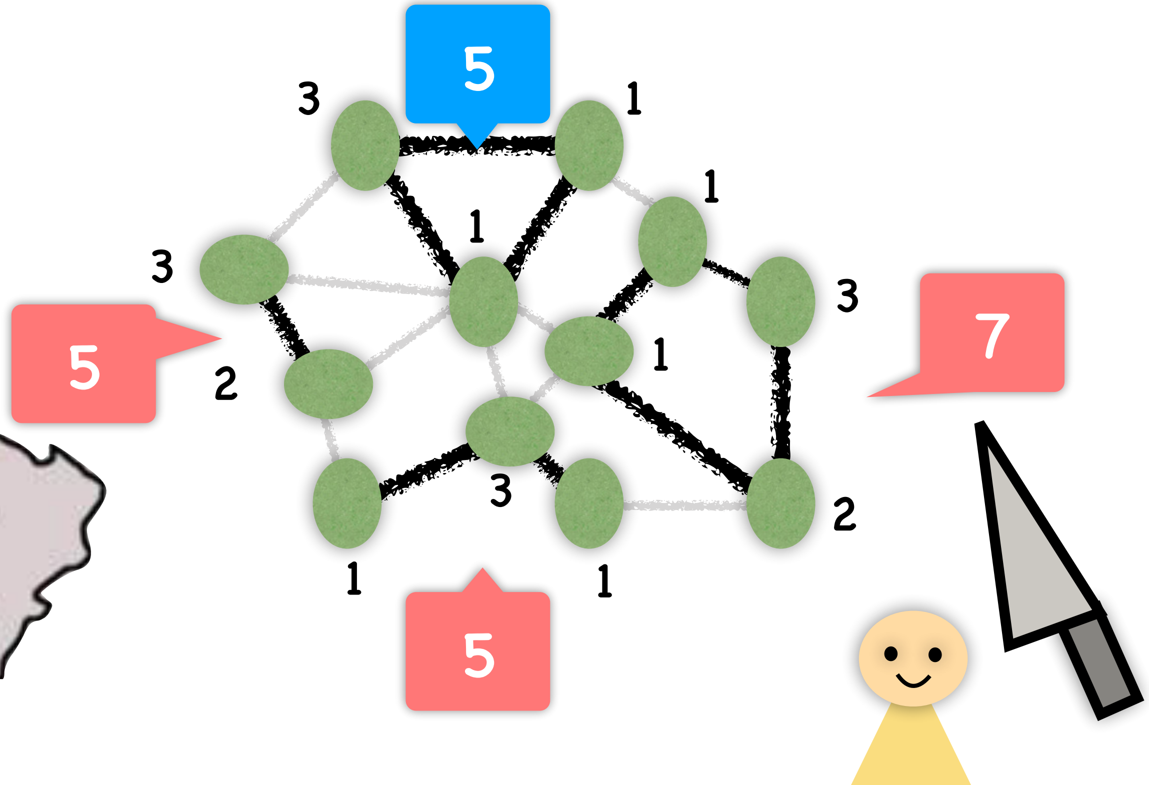
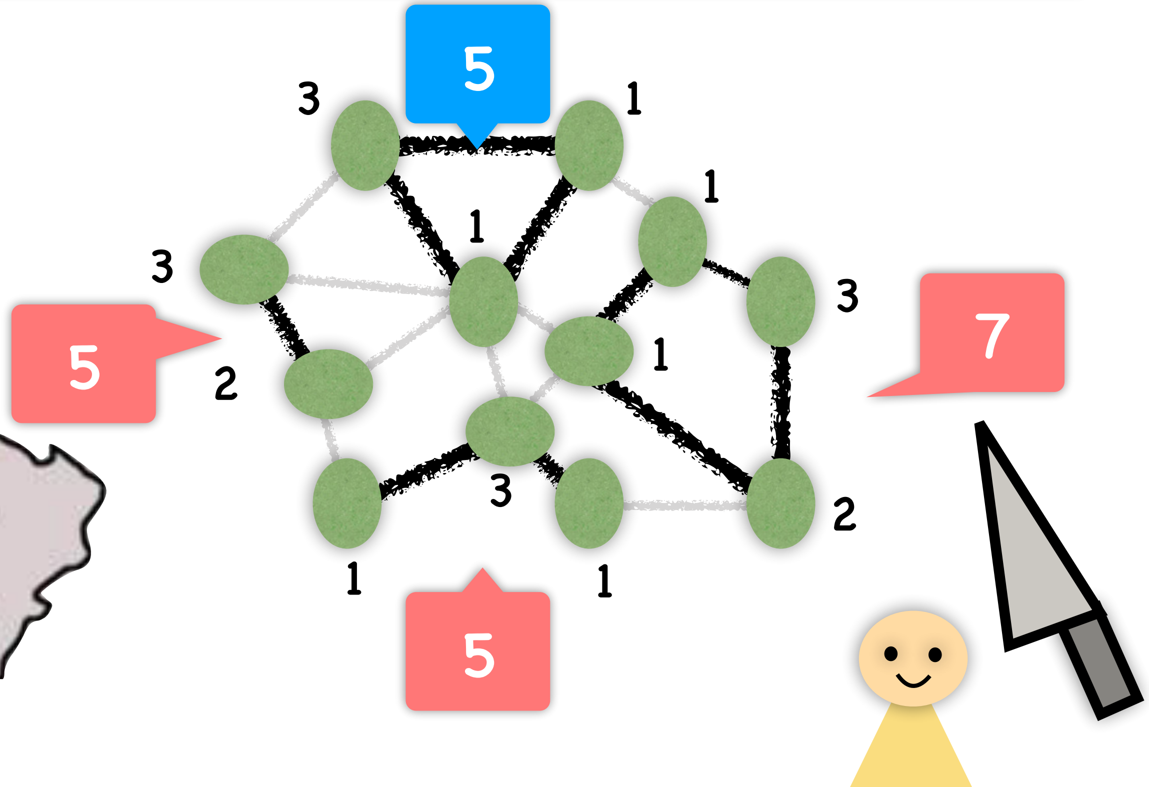


Maximin share

Maximin share (MMS) : u_i(i's bundle) $\geqslant$ MMS_i for all i in N

$MMS_i = \max\{\min_j u_i(P_j) \mid P_1,\dots,P_n$: a connected partition of G $\}$

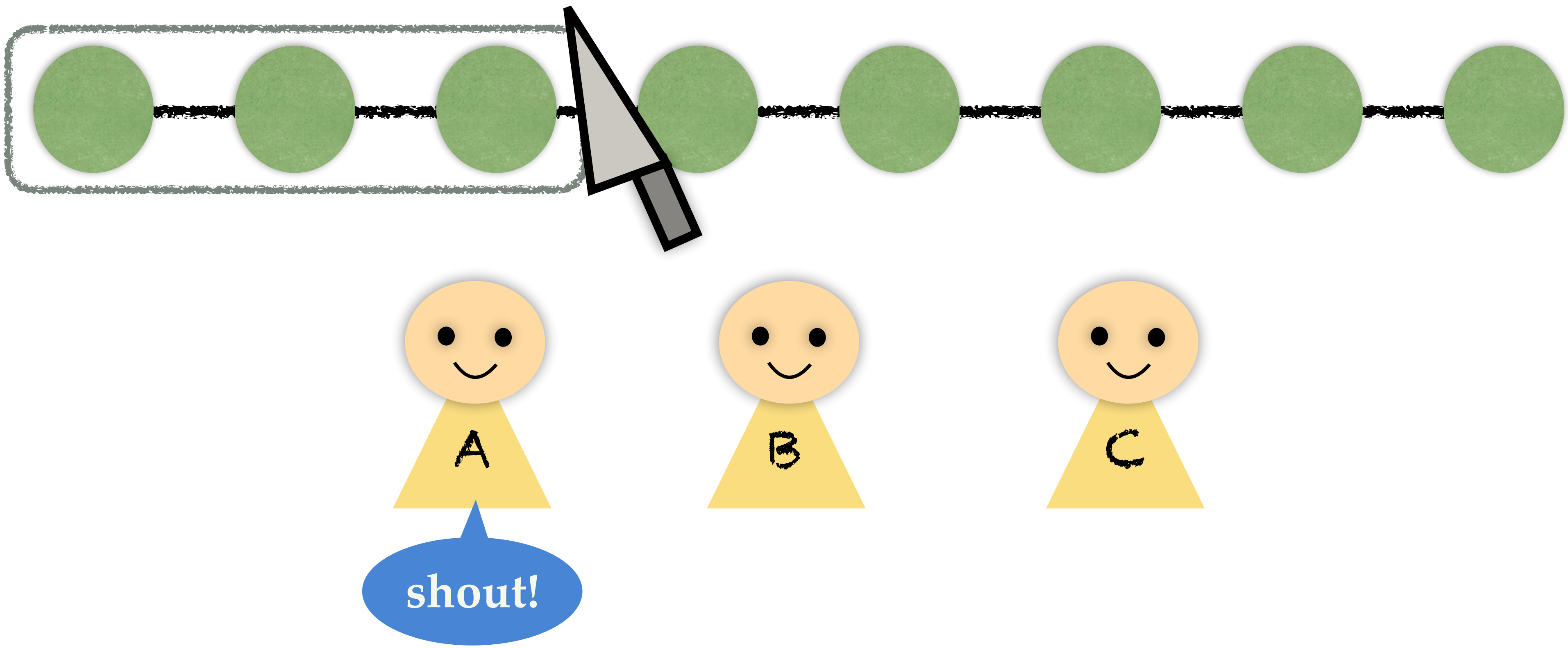


Unrestricted setting: MMS

- Identified special condition on the existence of MMS. Extensive experiments did not find any counter example [Bouveret and Lemaître, 2014].
- Intricate counter example with a number of goods exponential in the number of players [Procaccia and Wang, 2014]
- Reduced the number of goods to linear in the number of players [Kurokawa et al., 2016].

Moving-knife algorithm [Bouveret et al. 2017]

- Moving-knife procedures that achieve proportionality in cake-cutting produce MMS, when the graph is a path.



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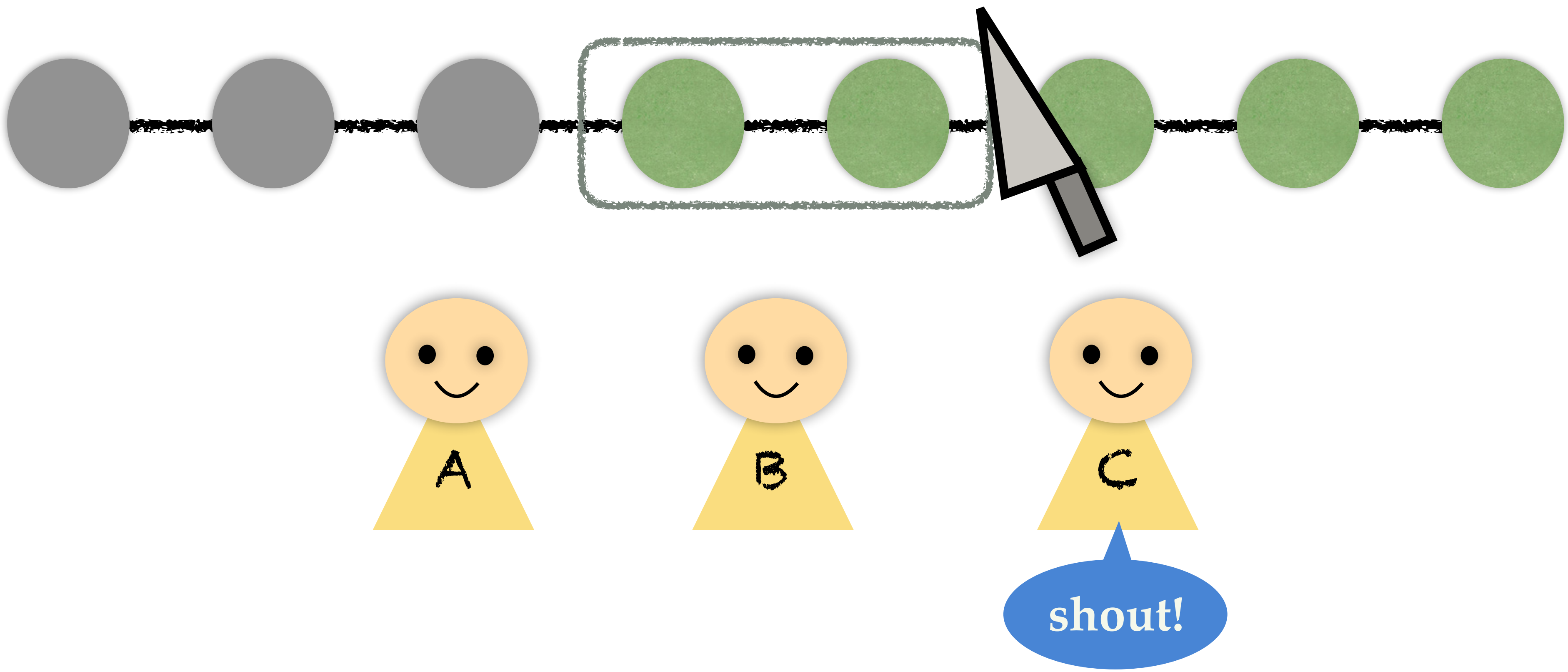

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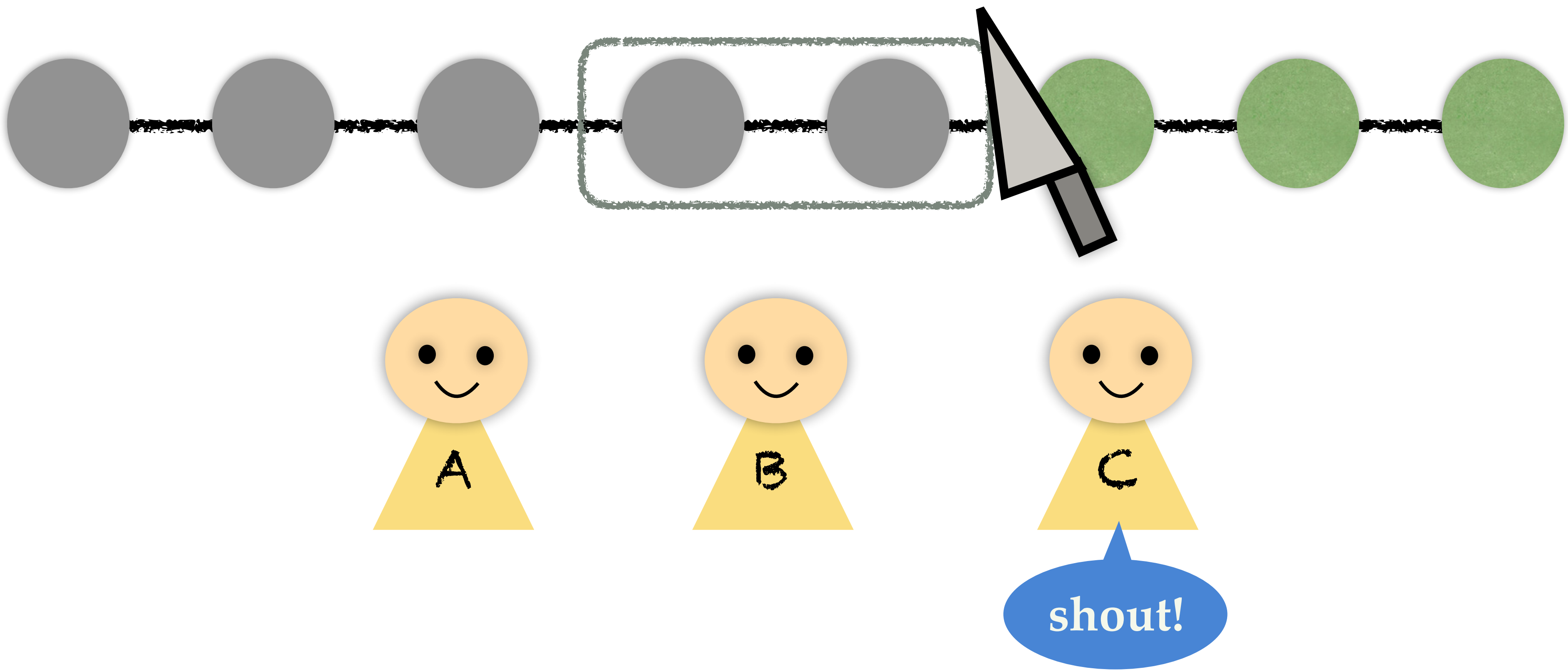

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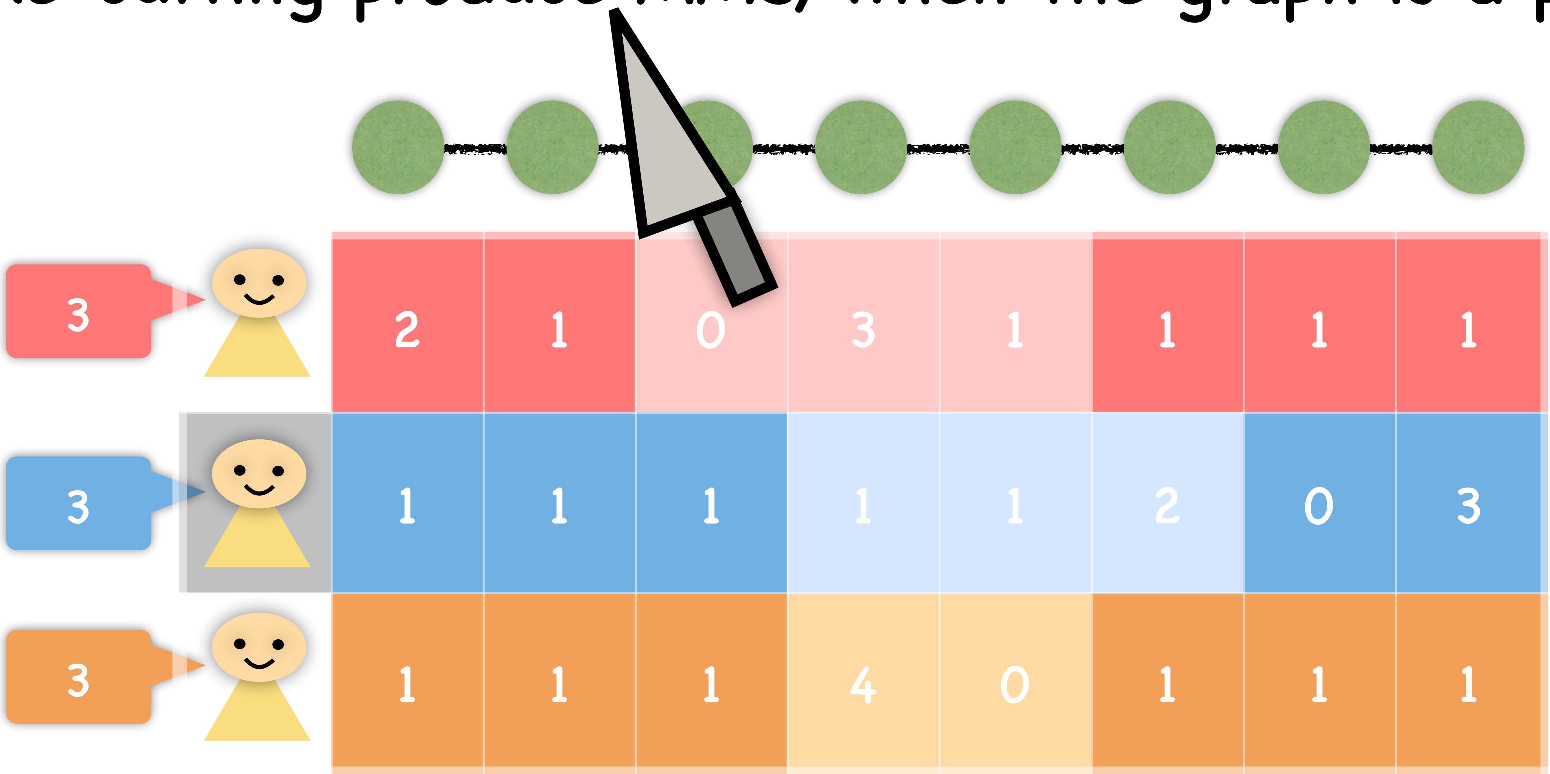

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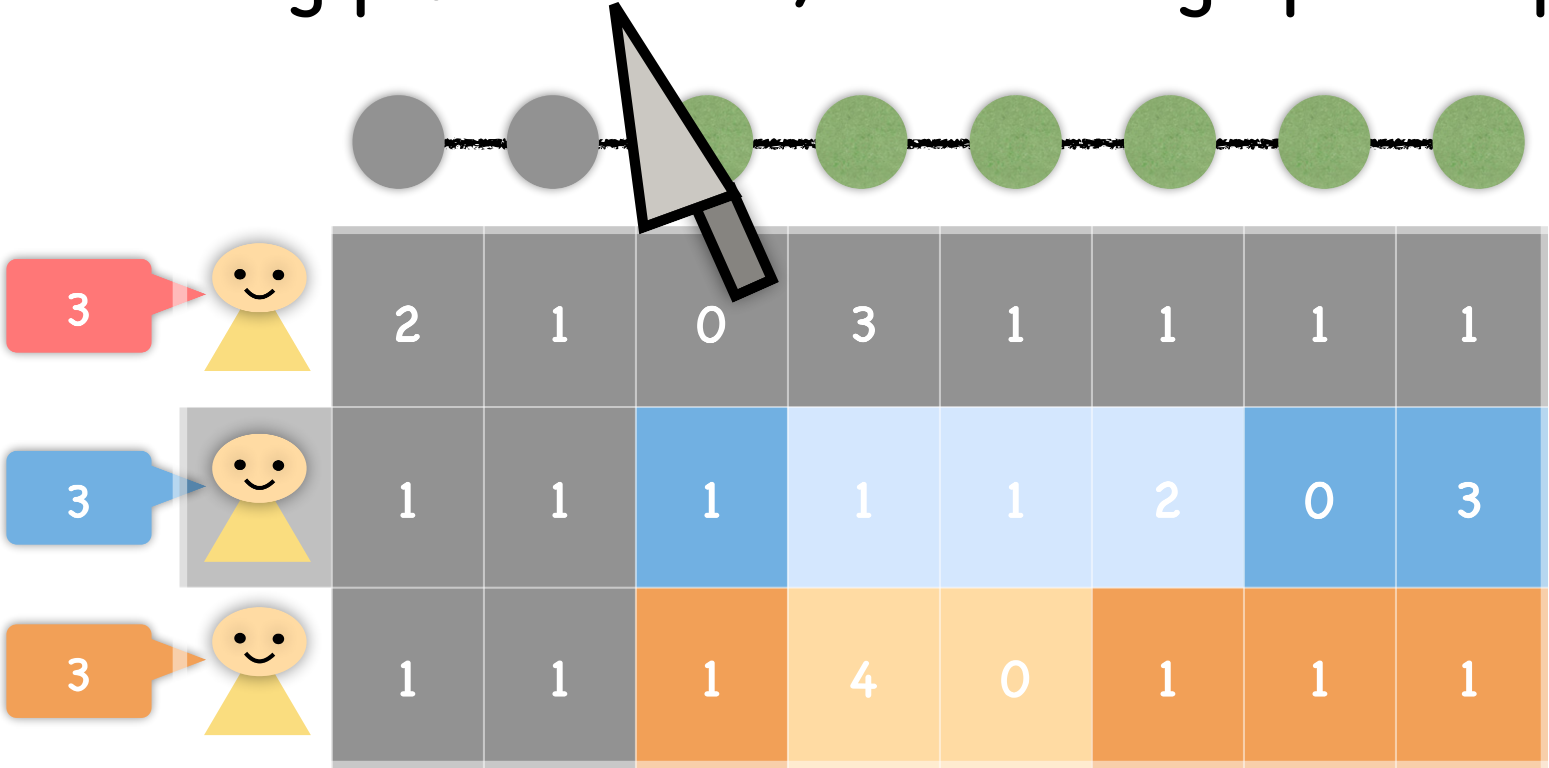
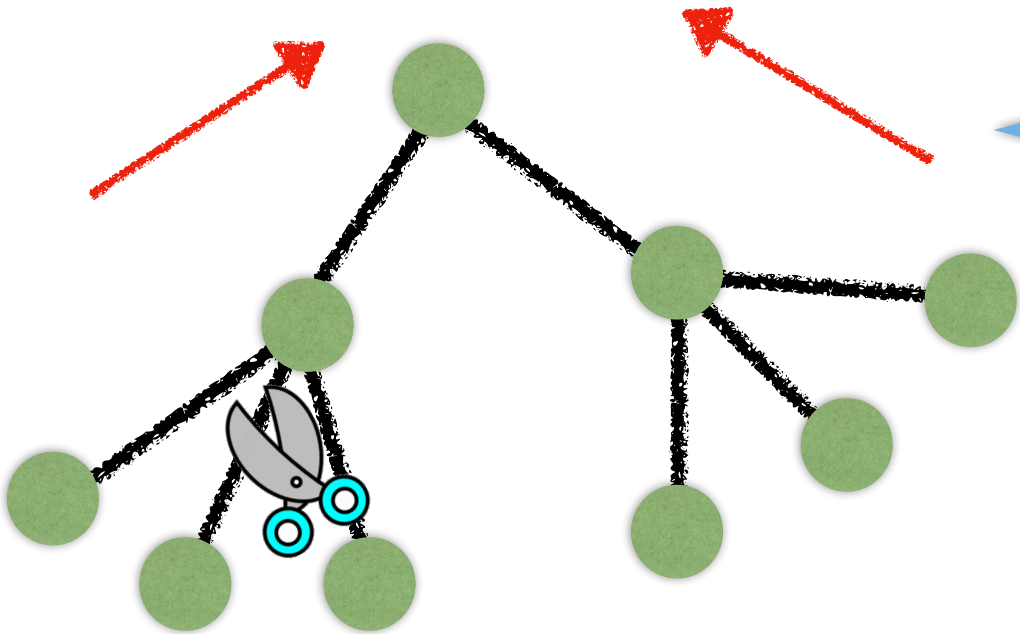


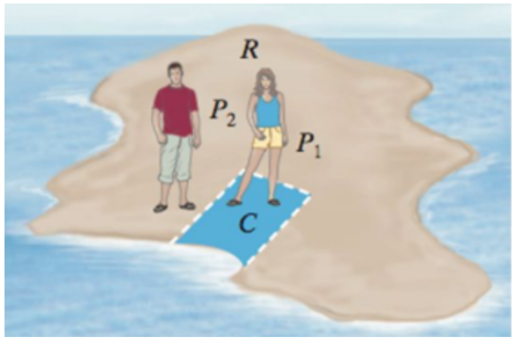
Maximin share for the reduced instance does not decrease.

MMS existence

- Theorem [Bouveret et al. 2017] MMS exists on trees and can be computed in polynomial time.



Discrete version of last diminisher



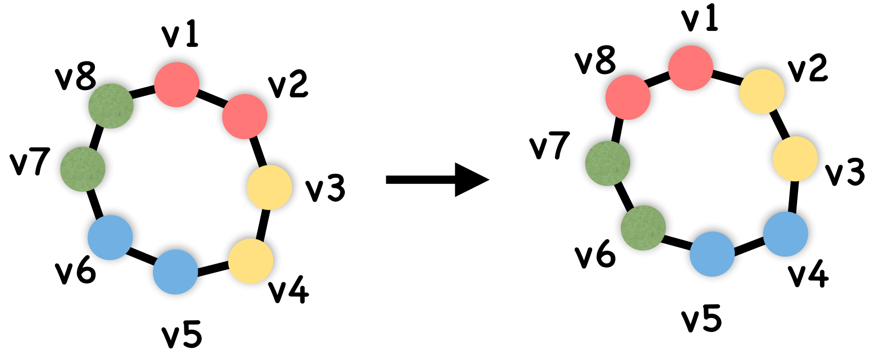
Cut a minimal subtree guaranteeing MMS for some player.

→ Recurse on the remaining instance.

MMS existence

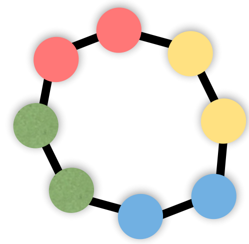
- Theorem [Bouveret et al. 2017] MMS may not exist on a single cycle of 8 vertices with 4 players.

	v1	v2	v3	v4	v5	v6	v7	v8
A, B	1	4	4	1	3	2	2	3
C, D	4	4	1	3	2	2	3	1



Create two types of players whose MMS partitions intersect with each other.

MMS: other work



- Lonc and Truszczynski [2018]:
  1/2-approximation for MMS in the case of cycles.

- Igarashi and Peters [2019]:
  A connected allocation satisfying MMS and Pareto-optimality exists when the graph is a tree.

  - NP-hard to compute even with binary additive valuations and even on a path.
    → polytime solvable for non-nested valuations.

MMS: open questions

- Complete characterisation of graphs guaranteeing MMS.

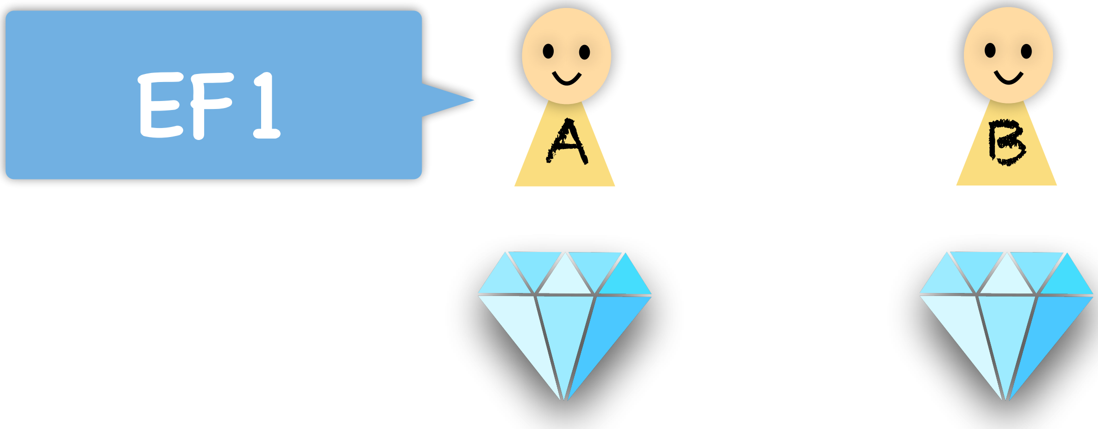
- The complexity of deciding the existence of a connected MMS.
  - Checking whether a given allocation is MMS is polytime solvable for a cycle.

- Existence of a connected MMS allocation of goods and bads.
  - Related works [Aziz et al., 2019; Bouveret et al. 2019]

Envy-freeness up to one good

- Envy-freeness need not exist —> Relaxations?
- Budish [2011]: Envy-freeness up to one good
  - For each i,j in N there is a good o^* in j's bundle with

$$u_i(\text{ i's bundle }) \geq u_i(\text{ j's bundle} \setminus \{o^*\}\text{ })$$



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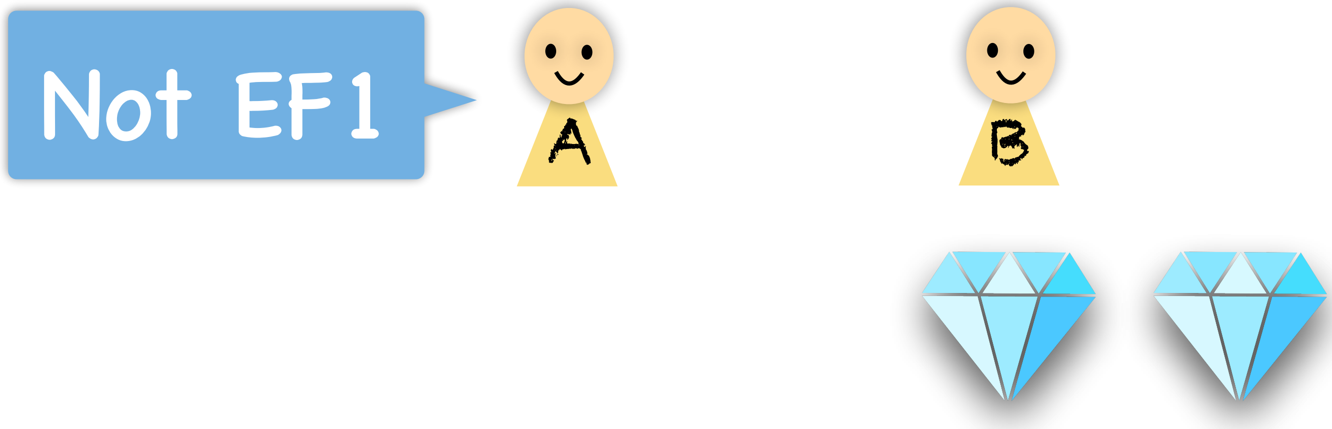


Envy-freeness up to one good

- Without connectivity constraints, EF1 always exists
  - Envy-graph algorithm [Lipton et al., 2004]
  - Round-robin procedure [Caragiannis et al., 2016]
  - Maximum Nash welfare [Caragiannis et al., 2016]

Theorem [Bilò et al. 2019; (a) and (d) appear also in Oh et al. 2019].
EF1 exists on a path

(a) when there are 2 agents (cut-and-choose); or

(b) when there are 3 agents (Stromquist's procedure); or

(c) when there are 4 agents (Sperner's lemma); or

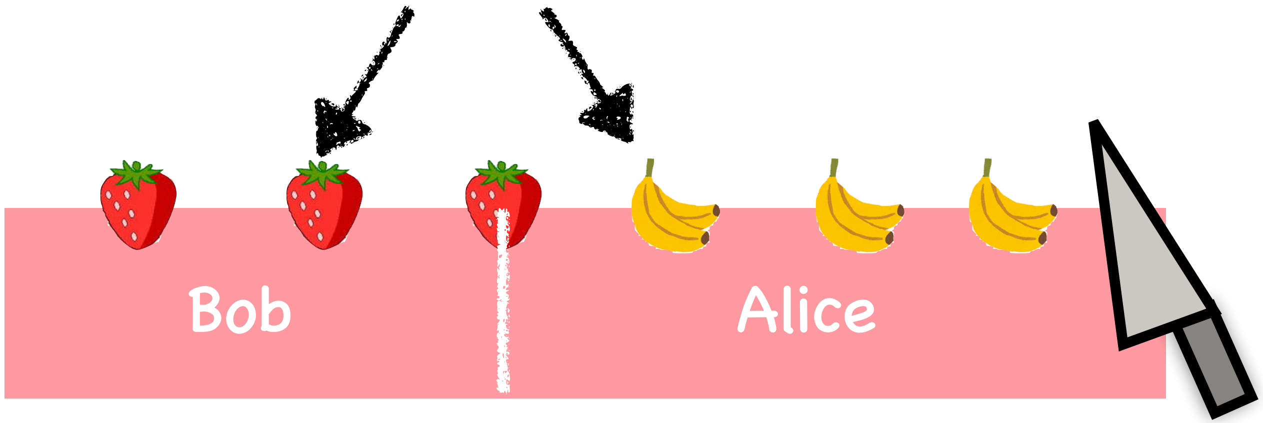
(d) when valuations are identical (≈ leximin)

EF1 for two agents

- Discrete version of cut and choose protocol

Divisible cake:

2. Bob chooses preferred piece and receives it



1. Alice divides the cake into two equally-valued pieces

3. Alice receives other piece

EF1 for two agents

- Discrete version of cut and choose protocol

v is an agent i's lumpy tie if

and

EF1 for two agents

- Discrete version of cut and choose protocol

- Alice selects her lumpy tie v and hides it

- Bob selects either the left or right piece

- Alice receives v and the remaining piece

- This is EF1. This works for all graphs with Hamiltonian path — any others?

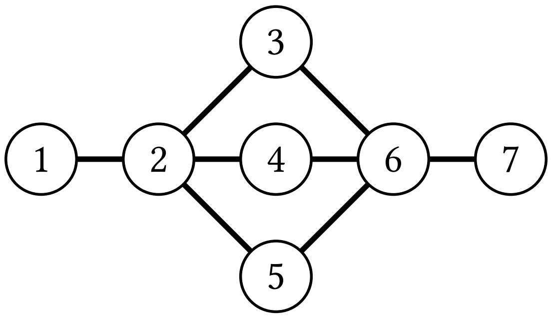
EF1 for two agents

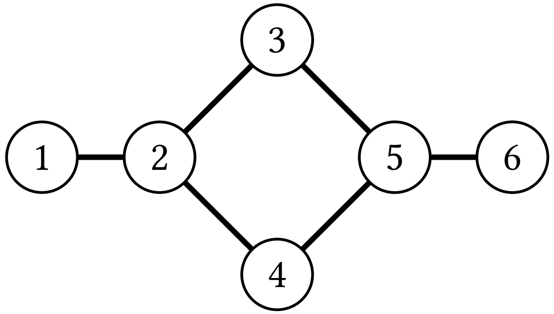
- Theorem [Bilò et al. 2019]

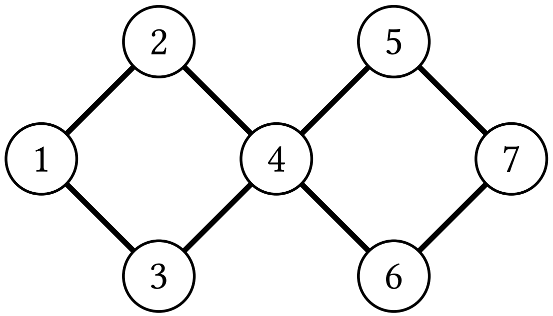
  For every connected graph G, the followings are equivalent:

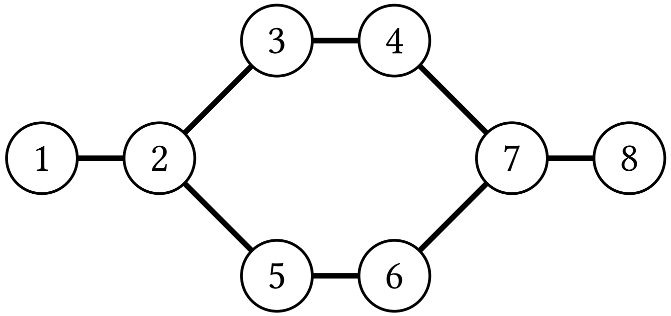
  (1) G admits a bipolar numbering.

  (2) G guarantees EF1 for two agents.









EF1 for more agents.

Theorem [Bilò et al. 2019; (a) and (d) appear also in Oh et al. 2019].
EF1 exists on a path

(a) when there are 2 agents (cut-and-choose); or

(b) when there are 3 agents (Stromquist's procedure); or

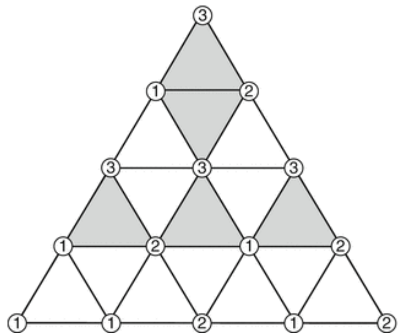
(c) when there are 4 agents (Sperner's lemma); or

(d) when valuations are identical (≈ leximin)

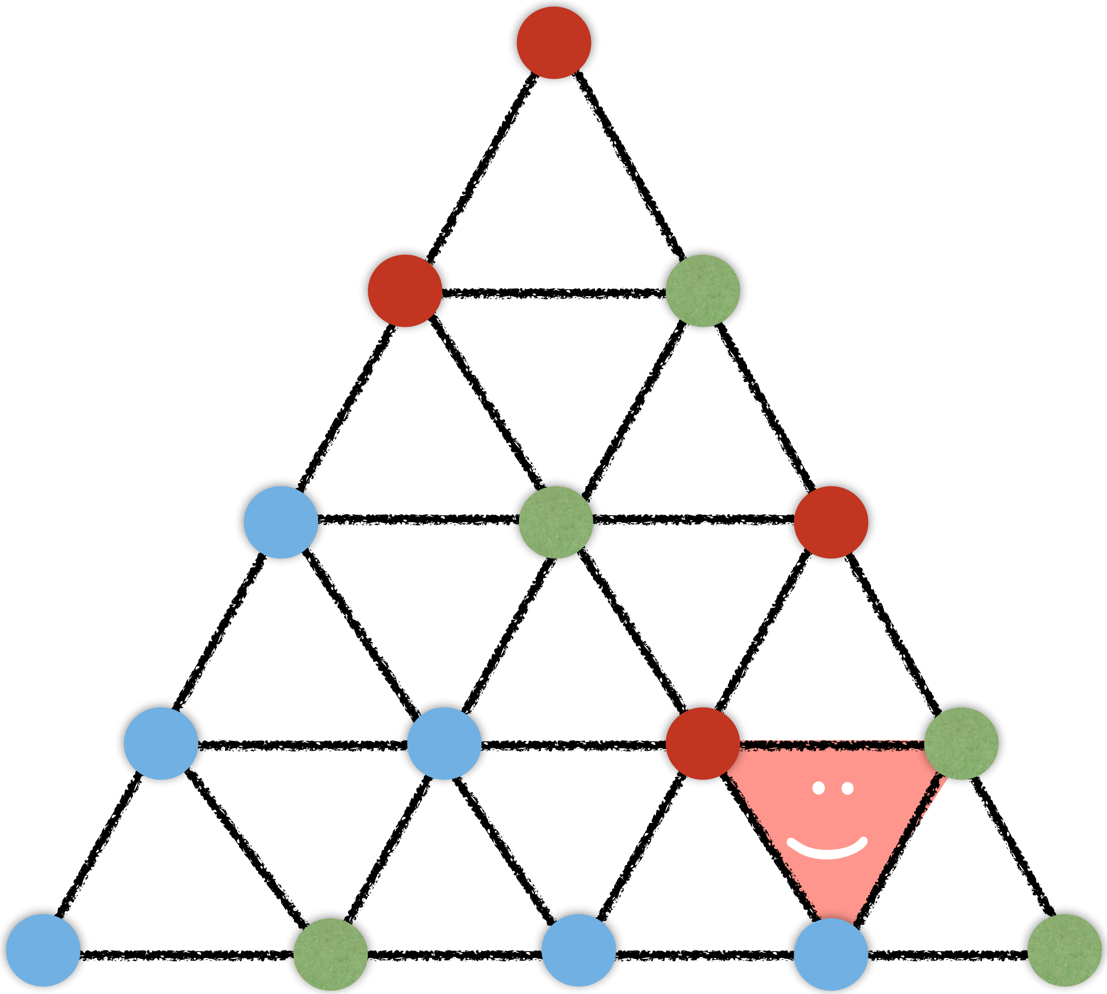
- By Sperner's lemma, EF2 always exists [Bilò et al. 2019]
- Existence extends to graphs with Hamiltonian path
- Existence does not require additive valuations

Sperner's Lemma

A combinatorial analog of the Brouwer/Kakutani fixed point theorem



Sperner's Lemma

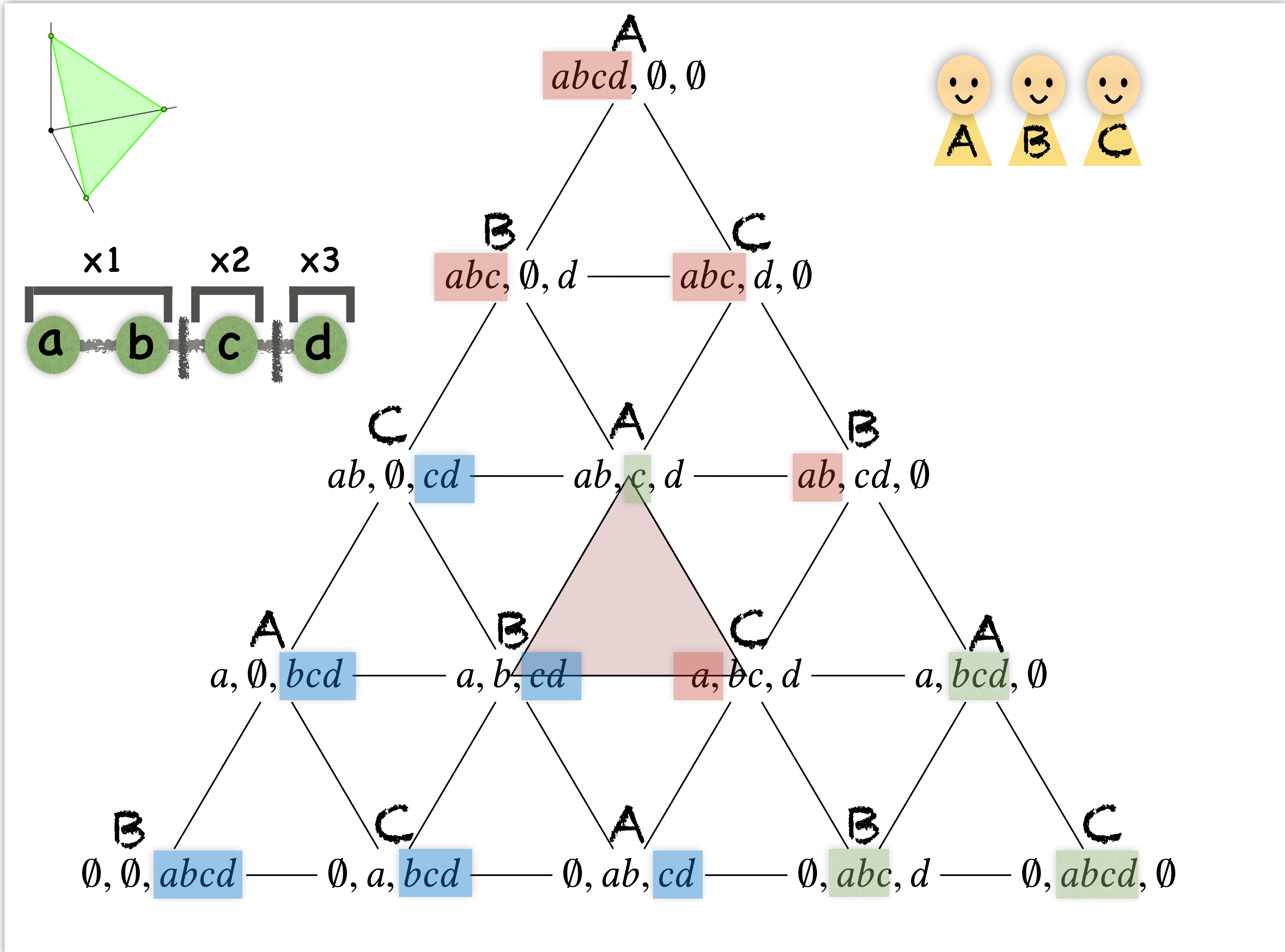


<u>Sperner's Lemma</u>

1. Color the corners with distinct colors.
2. Color every vertex of edge with the two colors of the endpoints.

→ a colorful triangle.

Sperner's Lemma and EF1



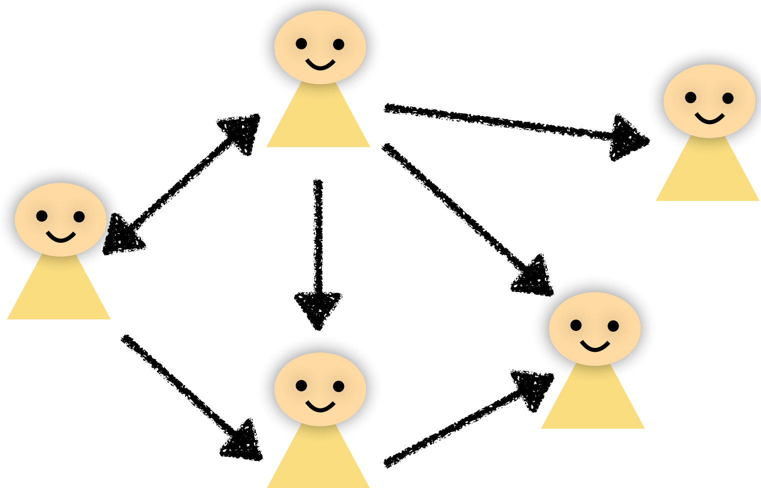
EF1: open question

- Characterisation of graphs that guarantee EF1 allocation beyond 2 agents.
- The complexity of finding an EF1 allocation with binary additive valuations.
- The complexity of finding an EF2 allocation.

Fair division over a social network

- Envy-freeness requires that no agent envies **any** other agent. In many situations, agents often do not even know each other.
- Local envy-freeness [Abebe et al. 2017, Bei et al. 2017] : No agent envies his/her neighbor in a social network.

Graphs represent envy-relations.



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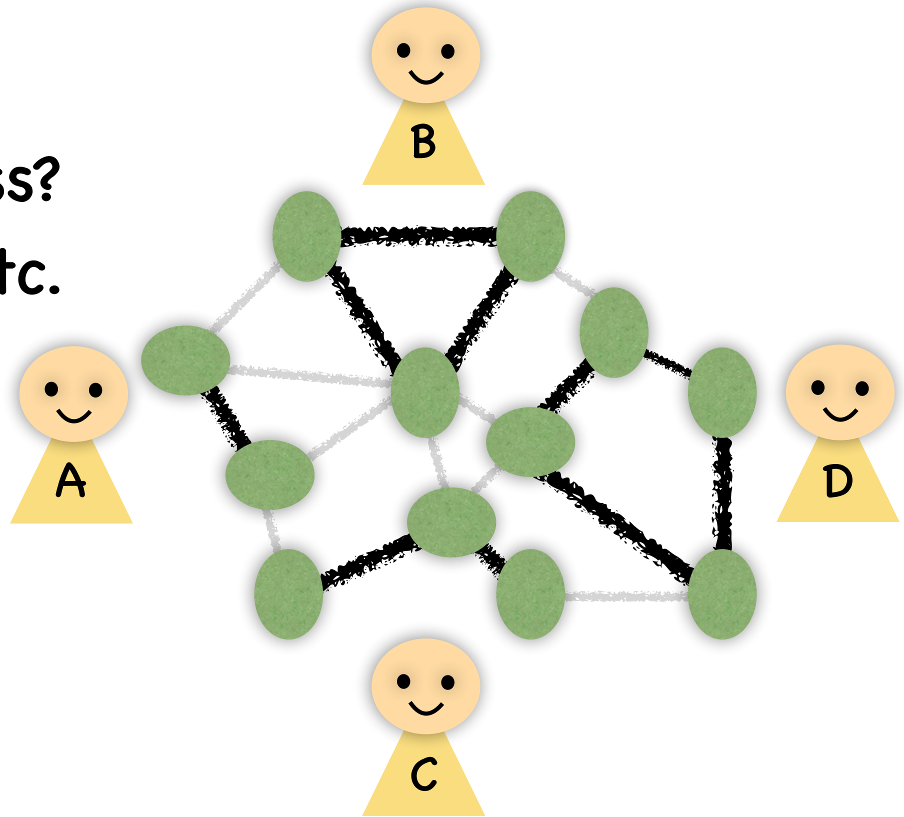
- Divisible items [Abebe et al. 2017, Bei et al. 2017]

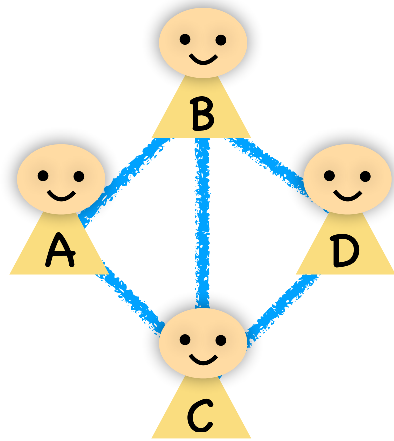
- Indivisible items [Bredereck et al., 2018]

Local fairness

- Combination of two models?
  Given a connected allocation, one can induce a social network.

- Local fairness?
  EF1, MMS, etc.





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