Margin of Victory for Tournament Solutions*

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Abstract

Tournament solutions are frequently used to select winners from a set of alternatives based on pairwise comparisons between them. Prior work has shown that several common tournament solutions tend to select large winner sets and therefore have low discriminative power. In this paper, we propose a general framework for refining tournament solutions. In order to distinguish between winning alternatives, and also between non-winning ones, we introduce the notion of *margin of victory* (MoV) for tournament solutions. MoV is a robustness measure for individual alternatives: For winners, the MoV captures the distance from dropping out of the winner set, and for non-winners, the distance from entering the set. In each case, distance is measured in terms of which pairwise comparisons would have to be reversed in order to achieve the desired outcome. For common tournament solutions, including the top cycle, the uncovered set, and the Banks set, we determine the complexity of computing the MoV and provide bounds on the MoV for both winners and non-winners. We then reveal a number of structural insights on the MoV by investigating fundamental properties such as monotonicity and consistency with respect to the covering relation. Furthermore, we provide experimental evidence on the extent to which the MoV notion refines winner sets in tournaments generated according to various stochastic models. Our results can also be viewed from the perspective of bribery and manipulation.

1 Introduction

Tournaments serve as a practical tool for modeling scenarios involving a set of alternatives along with pairwise comparisons between them. Perhaps the most common example of a tournament is a round-robin sports competition, where every pair of teams play each other once and there is no tie in match outcomes. Another application, typical especially in the social choice literature, concerns elections: here, alternatives represent election candidates, and pairwise comparisons capture the majority relation between pairs of candidates. In order to select the set of "winners" from a tournament, several methods, known in the literature as *tournament solutions*, have been proposed. Given the ubiquity of tournaments, it is hardly surprising that tournament solutions have drawn substantial interest from researchers in the past few decades, including but not limited to those in the artificial intelligence community (e.g., Laslier, 1997; Woeginger, 2003; Hudry, 2009; Brandt et al., 2011; Brandt et al., 2014; Aziz et al., 2015; Mnich et al., 2015; Ramamohan et al., 2016; Dey, 2017; Brandt et al., 2018; Han and van Deemen, 2019).

Although tournament solutions provide a rich supply of procedures for choosing tournament winners according to various criteria, they often exhibit low discriminative power because the chosen winner sets tend to be large. Indeed, previous work has shown that common tournament solutions such as the top cycle, the uncovered set, the Banks set, and the minimal covering set almost never exclude any

*This paper unifies and expands earlier versions appearing in the *Proceedings of the 34th AAAI Conference on Artificial Intelligence* [Brill et al., 2020] and the *Proceedings of the 35th AAAI Conference on Artificial Intelligence* [Brill et al., 2021].

alternative in a random tournament [Fey, 2008, Scott and Fey, 2012], while the bipartisan set includes on average half of the alternatives in the winner set [Fisher and Ryan, 1995].[1,2] Given that the purpose of tournament solutions is to distinguish between the alternatives, this naturally raises the question of how tournament solutions can be refined in order to differentiate among the winners of a given tournament.

In this paper, we propose a general framework for refining tournament solutions and for distinguishing among the winners—as well as among the non-winners—of a tournament. We introduce the concept of *margin of victory* (MoV) for tournament solutions, which captures how close a winner is to dropping out of the winner set, and by symmetry how close a non-winner is to entering the winner set. As we discuss in Section 1.2, the concept of MoV is not new, but this is the first time it has been applied to tournament solutions, to the best of our knowledge. For a given tournament and weights on the tournament edges, we define the MoV of a winner as the minimum total weight of edges whose reversals take it out of the winner set. Analogously, the MoV of a non-winner is defined as the negative of the minimum total weight of edges whose reversals bring it into the winner set. An important special case is when the edges are unweighted: in this case, the problem reduces to finding the minimum *number* of edges to be reversed in order to take a winner out of the winner set or bring a non-winner into it.

The edge weights in our MoV framework can be interpreted in a number of different ways. Generally speaking, they represent the strength of the edges or the cost that one incurs by reversing them. In an election, a weight may reflect the proportion of voters who agree with the corresponding pairwise comparison, while in a sports competition, it may indicate the gap between the two teams in the match result. Alternatively, our refinements can also be viewed through the lens of bribery and manipulation, another topic of recent interest in the artificial intelligence community (see, e.g., the book chapter by Faliszewski and Rothe [2016]). In this context, the weights express the amount of bribe that a manipulator needs to pay in order to reverse a pairwise comparison; the recipients of the bribe are voters in the case of an election and teams or referees in the case of a sports competition. While the MoV for winners is useful for refining tournament solutions, the MoV for non-winners is more relevant in the context of bribery and manipulation, as the desired goal is often to ensure that a certain alternative is a winner.

1.1 Our Results

We investigate a wide range of aspects of the MoV notion with respect to four common tournament solutions—the Copeland set (CO), the top cycle (TC), the uncovered set (UC), and the Banks set (BA)—as well as k-kings, a natural class of solutions that lie between the top cycle and the uncovered set. The definitions of these tournament solutions, as well as other formal definitions and notations, can be found in Section 2.

We begin in Section 3 by determining the complexity of computing the MoV for both winners and non-winners for each tournament solution, in both the unweighted and weighted settings. For winners, we show that the problem can be solved in polynomial time for CO, TC, and UC but is NP-hard for BA, whereas for k-kings, we demonstrate an interesting distinction between the cases $k = 3$ and $k \geq 4$: the former case is tractable even in the weighted setting, while the latter is already intractable in the unweighted setting. On the other hand, although the same complexity results hold for non-winners as for winners with respect to CO, TC, and BA, we exhibit differences concerning UC and k-kings. For these latter tournament solutions, computing the MoV for non-winners is NP-hard in the weighted setting but can be done in subexponential time in the unweighted setting.

In the remaining sections, we focus on the unweighted setting, starting in Section 4 with bounds on the MoV. For each tournament solution, we derive tight or asymptotically tight lower and upper bounds,

[1] These results assume that tournaments are chosen from the uniform distribution. Brandt and Seedig [2016] and Saile and Suksompong [2020] relaxed this assumption and studied the discriminative power of tournament solutions when tournaments are generated according to different stochastic models.

[2] Brandt et al. [2018] showed that any tournament solution satisfying the property of *stability*, including the top cycle and the bipartisan set, chooses at least half of the alternatives on average.

	MoV for Winners		MoV for Non-Winners	
	unweighted	weighted	unweighted	weighted
Copeland (CO)	P (3.1)	P (3.1)	P (3.6)	P (3.6)
Top Cycle (TC)	P (3.3)	P (3.3)	P (3.10)	P (3.10)
Uncovered Set (UC)	P (3.3)	P (3.3)	$n^{O(\log n)}$ (3.7)	NP-hard (3.8)
3-kings	P (3.3)	P (3.3)	P (4.2)	NP-hard (3.9)
k-kings (for $k \geq 4$)	NP-hard (3.4)	NP-hard (3.4)	P (4.2)	NP-hard (3.9)
Banks set (BA)	NP-hard (3.5)	NP-hard (3.5)	NP-hard (3.11)	NP-hard (3.11)

	Bounds on MoV	
	lower bound	upper bound
Copeland (CO)	$-(n-2)$ (4.3)	$\lfloor n/2 \rfloor$ (4.1)
Top Cycle (TC)	-1 (4.2)	$\lfloor n/2 \rfloor$ (4.1)
Uncovered Set (UC)	$-\lceil \log_2 n \rceil$ (4.4)	$\lfloor n/2 \rfloor$ (4.1)
k-kings (for $k \geq 3$)	-1 (4.2)	$\lfloor n/2 \rfloor$ (4.1)
Banks set (BA)	$-\lceil \log_2 n \rceil$ (4.4)	$\lfloor n/2 \rfloor$ (4.1)

Table 1: Overview of our results in Sections 3 and 4, with n denoting the number of alternatives in the tournament. The computational results for Copeland (first row of top table) also follow from Faliszewski et al. [2009]; for completeness, we give proofs tailored to our setting. The numbers in parentheses refer to the corresponding theorem or corollary numbers.

which tell us how many pairwise comparisons we may need to reverse in order to bring an alternative into or take it out of the winner set. In particular, for all tournament solutions, the MoV for winners can be as high as $\lfloor n/2 \rfloor$ but no higher, where n denotes the number of alternatives in the tournament. Lower bounds on the MoV value of non-winners, on the other hand, depend on the tournament solution in question: Turning a non-winner into a winner may require a linear number of reversals for CO and a logarithmic number for UC and BA, while a single edge reversal is always sufficient for TC and k-kings. Our results in Sections 3 and 4 are summarized in Table 1.

Next, in Section 5, we turn to the axiomatic approach and examine structural properties of the MoV. We define consistency axioms with respect to two important features of tournaments: the covering relation and the outdegrees of the alternatives. We prove that the MoV values of all considered tournament solutions are consistent with the covering relation, thereby showing that our notion is aligned with an important strength indicator of the alternatives. In order to establish this result, we introduce a new property for tournament solutions called *transfer-monotonicity*, which may be of independent interest. On the other hand, we demonstrate that only TC and CO have MoV functions that exhibit some form of consistency in view of the outdegrees (a.k.a. *Copeland scores*) of the alternatives. We argue that this can be viewed as a positive result, since it shows that several tournament solutions take more structure of the tournament into account than simply the outdegrees. Additionally, we present a simple formula for the MoV of TC and k-kings for $k \geq 4$ which holds with high probability; this implies that even though the computation problem for k-kings is NP-hard (as we show in Section 3), an efficient heuristic exists.

Finally, in order to better understand how the MoV of various tournament solutions behave in practice, we conduct computer experiments using tournaments generated randomly according to six well-studied stochastic models. Our results and analyses, along with a link to the code for our implementation, can be found in Section 6.

1.2 Further Related Work

Despite their origins in social choice theory, tournament solutions have found applications in a large number of areas including game theory [Fisher and Ryan, 1995], webpage ranking [Brandt and Fischer, 2007], dueling bandit problems [Ramamohan et al., 2016], and philosophical decision theory [Podgorski, 2020]. As is the case for social choice theory in general, early studies of tournament solutions were primarily based on the axiomatic approach. With the rise of computational social choice in the past fifteen years or so, tournament solutions have also been thoroughly examined from an algorithmic perspective. For an overview of the literature, we refer the reader to the surveys by Laslier [1997], Hudry [2009], Brandt et al. [2016a], and Suksompong [2021].

While our work is the first to consider a MoV concept for tournament solutions (to the best of our knowledge), a related notion with the same name has been extensively explored in the context of voting. Unlike in our setting, where the MoV serves the purpose of distinguishing among alternatives, in voting the MoV is typically used to measure the robustness of election outcomes [Cary, 2011, Magrino et al., 2011, Xia, 2012, Dey and Narahari, 2015]. As such, the notion there is defined for election outcomes as a whole rather than for individual alternatives; the same holds for the robustness measure of Shiryaev et al. [2013]. MoV continues to be a popular concept in recent research, for example in the context of sports modeling [Kovalchik, 2020], election control [Castiglioni et al., 2020], and political and educational districting [Stoica et al., 2020, Boehmer et al., 2021].

A long line of work has investigated various forms of bribery and manipulation in tournaments. This includes manipulating the tournament bracket to help a certain candidate win the tournament [Vu et al., 2009, Vassilevska Williams, 2010, Chatterjee et al., 2016, Kim et al., 2017, Ramanujan and Szeider, 2017, Aziz et al., 2018, Gupta et al., 2019] and bribing players to lose matches intentionally [Russell and Walsh, 2009, Kim and Vassilevska Williams, 2015, Mattei et al., 2015, Konicki and Vassilevska Williams, 2019]. In particular, Russell and Walsh [2009] considered a model where only a given subset of edges can be reversed while other edges are assumed to be fixed—this constitutes a special case of our weighted setting, with sufficiently high weights on fixed edges. In the context of bribery in voting, Faliszewski et al. [2009] considered a "microbribery" setting in which voters can be bribed to change individual pairwise comparisons between candidates, even if this results in intransitive preferences of the voter. This corresponds to our weighted setting, with weights given by pairwise majority margins.

A closely related problem is finding possible (resp., necessary) winners of partially specified tournaments: Given a tournament with some missing edges, the goal is to determine whether a certain alternative can be a winner for some (resp., all) completions of the tournament [Aziz et al., 2015]. We observe that both variants can be reduced to computing the MoV in the weighted setting, by considering an arbitrary completion of the partial tournament and making the original edges prohibitively expensive to reverse. Yang and Guo [2017] studied this setting from a parameterized complexity perspective; one of their results (Theorem 3 in their paper) addresses a problem equivalent to the decision version of computing the MoV with respect to the uncovered set.

Finally, a different framework for refining tournament solutions has been proposed by Kruger and Airiau [2017]. Specifically, they considered refinements of tournament solutions based on their *binary tree representations*. Their approach can only be applied to solutions that admit such a representation, and moreover, different representations may yield different refinements. In contrast, our MoV framework can be used for arbitrary tournament solutions and does not depend on representation issues.

2 Preliminaries

A *tournament* $T = (V, E)$ is a directed graph such that there is exactly one directed edge between every pair of vertices. The vertices of a tournament T, denoted $V(T)$, are often referred to as *alternatives* or *nodes*. Let $n = |V(T)|$. The set of directed edges of T, denoted $E(T)$, represents an asymmetric and connex *dominance relation* on the set of alternatives. An alternative x is said to *dominate* another alternative y if $(x, y) \in E(T)$ (i.e., there is a directed edge from x to y). When the tournament is clear from the context, we often write $x \succ y$ to denote $(x, y) \in E(T)$. By definition, for each pair x, y of distinct alternatives, either x dominates y ($x \succ y$) or y dominates x ($y \succ x$), but not both. The dominance relation can be extended to sets by writing $X \succ Y$ if $x \succ y$ for all $x \in X$ and $y \in Y$. A set $X \subseteq V(T)$ is called a *dominating set* in T if every alternative outside of X is dominated by at least one alternative in X.

For a given tournament T and $x \in V(T)$, the *dominion* of x, denoted by $D(x)$, is defined as the set of alternatives y such that $x \succ y$. Similarly, the set of *dominators* of x, denoted by $\overline{D}(x)$, is defined as the set of alternatives y such that $y \succ x$. The *outdegree* of x is denoted by $\text{outdeg}(x) = |D(x)|$, and the *indegree* of x by $\text{indeg}(x) = |\overline{D}(x)|$. For any $x \in V(T)$, it holds that $\text{outdeg}(x) + \text{indeg}(x) = n - 1$. An alternative $x \in V(T)$ is said to be a *Condorcet winner* in T if it dominates every other alternative (i.e., $\text{outdeg}(x) = n - 1$), and a *Condorcet loser* in T if it is dominated by every other alternative (i.e., $\text{outdeg}(x) = 0$). See Figure 1 for an example tournament. A tournament is called *regular* if all of its alternatives have the same outdegree. A regular tournament exists for every odd size, but not for any even size.

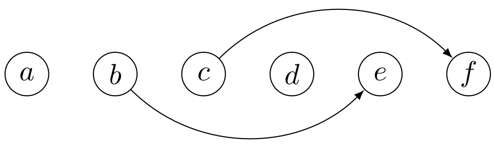

Figure 1: Tournament T with $V(T) = \{a, b, c, d, e, f\}$. All omitted edges are assumed to point from right to left (e.g., $D(f) = \{a, b, d, e\}$ and a is a Condorcet loser in T).

For $U \subseteq V(T)$, $T|_U$ denotes the restriction of T to U, and T_{-x} is short for $T|_{V(T)\setminus\{x\}}$. For an edge $e = (x, y)$, we let $\overline{e}$ denote its reversal, i.e., $\overline{e} = (y, x)$. Similarly, for a set of edges $R \subseteq E(T)$, we define $\overline{R} = \{\overline{e} : e \in R\}$.

2.1 Tournament Solutions

A *tournament solution* is a function that maps each tournament to a nonempty subset of its alternatives, usually called the set of *winners* or the *choice set*. A tournament solution must not distinguish between isomorphic tournaments; in particular, if there is an automorphism that maps an alternative x to another alternative y in the same tournament, any tournament solution must either choose both x and y or neither of them. The set of winners of a tournament T with respect to a tournament solution S is denoted by $S(T)$. The tournament solutions considered in this paper are as follows:

- The *Copeland set* (CO) is the set of alternatives with the largest outdegree.
- The *top cycle* (TC) is the (unique) smallest nonempty set B of alternatives such that $B \succ V(T) \setminus B$. Equivalently, TC is the set of alternatives that can reach every other alternative via a directed path.
- The *uncovered set* (UC), is the set of alternatives that are not "covered" by any other alternative. An alternative x is said to *cover* another alternative y if $D(y) \subseteq D(x)$. Equivalently, UC is the set of alternatives that can reach every other alternative via a directed path of length at most two.
- The set of *k-kings*, for an integer $k \geq 3$, is the set of alternatives that can reach every other alternative via a directed path of length at most k.

- The *Banks set* (BA) is the set of alternatives that appear as the Condorcet winner (i.e., maximal element) of some transitive subtournament that cannot be extended.[3]

All of these tournament solutions satisfy *Condorcet-consistency*, meaning that whenever a Condorcet winner exists, it is chosen as the unique winner. It is clear from the definitions that UC (the set of "2-kings") is contained in the set of k-kings for any $k \geq 3$, which is in turn a subset of TC (the set of "$(n-1)$-kings"). Moreover, both CO and BA are contained in UC [Laslier, 1997].

For an edge $e \in E(T)$, denote by T^e the tournament that results from T when reversing e. A tournament solution S is said to be *monotonic* if for any edge $e = (y, x) \in E(T)$,

$$x \in S(T) \quad \text{implies} \quad x \in S(T^e).$$

In other words, a tournament solution is monotonic if a winner remains in the choice set whenever its dominion is enlarged (while the dominion of no other alternative is enlarged). Equivalently, monotonicity means that a non-winner remains outside of the choice set whenever it becomes dominated by an additional alternative. All of the above tournament solutions are monotonic (see Proposition 5.4).

2.2 Margin of Victory

We now introduce the central notion of our paper. We define the margin of victory (MoV) for a winning (resp., non-winning) alternative in terms of sets of edges whose reversals result in the alternative becoming a non-winner (resp., winner). Edge sets with this property will be called destructive (resp., constructive) reversal sets. To formally define these concepts, we need additional notation. For a tournament T and a set $R \subseteq E(T)$ of edges, we let T^R denote the tournament that results from T when reversing all edges in R, i.e., $V(T^R) = V(T)$ and $E(T^R) = (E(T) \setminus R) \cup \overline{R}$.

Fix a tournament solution S and consider a tournament T. An edge set $R \subseteq E(T)$ is called a *destructive reversal set (DRS)* for $x \in S(T)$ if $x \notin S(T^R)$. Analogously, R is called a *constructive reversal set (CRS)* for $x \in V(T) \setminus S(T)$ if $x \in S(T^R)$.[4] In general, destructive and constructive reversal sets are not unique, and finding *some* DRS or CRS is usually easy. For example, for all Condorcet-consistent tournament solutions S, a straightforward CRS for an alternative $x \notin S(T)$ is given by $R = \{(y, x) : y \in \overline{D}(x)\}$. This is because x is a Condorcet winner in T^R.

We furthermore assume that we are given a weight function $w : E(T) \to \mathbb{R}_{>0}$ that assigns a positive weight $w(e) > 0$ to each edge $e \in E(T)$.[5] The weight of an edge can be thought of as the *cost* that is incurred by reversing the edge. The cost of a set $R \subseteq E(T)$ is $w(R) = \sum_{e \in R} w(e)$. A natural special case is the setting in which reversing is equally costly for all edges. In this *unweighted* setting, we assume $w(e) = 1$ for all $e \in E(T)$, and finding a minimum cost reversal set reduces to finding a reversal set of minimum cardinality.

We are ready to define the main concept of this paper.

Definition 2.1. For a tournament solution S, a tournament T, and a weight function $w : E(T) \to \mathbb{R}_{>0}$, the *margin of victory* of an alternative $x \in S(T)$ is given by

$$\mathsf{MoV}_S(x, T) = \min\{w(R) : R \text{ is a destructive reversal set for } x \text{ in } T\},$$

and for an alternative $x \in V(T) \setminus S(T)$, it is given by

$$\mathsf{MoV}_S(x, T) = -\min\{w(R) : R \text{ is a constructive reversal set for } x \text{ in } T\}.$$

[3] We say that an alternative $x \in V(T) \setminus V(T')$ *extends* a transitive subtournament T' if x dominates all alternatives in T'.

[4] The terms "destructive" and "constructive" are borrowed from the literature on control and bribery in voting (e.g., Faliszewski and Rothe, 2016), where the goal is either to prevent a given candidate from winning (destructive control/bribery) or to make a given candidate a winner (constructive control/bribery).

[5] We forbid zero-weight edges for technical reasons. Their existence can be imitated by setting their cost to a small $\epsilon > 0$.

	a	b	c	d	e	f
$\mathsf{MoV}_{UC}(x,T)$	-2	-1	1	1	1	2
minimum DRS/CRS	$\{(f,a),(d,a)\}$	$\{(f,b)\}$	$\{(c,f)\}$	$\{(d,c)\}$	$\{(e,d)\}$	$\{(f,e),(f,b)\}$

Table 2: MoV values and minimum reversal sets with respect to UC for the tournament T in Figure 1.

By definition, $\mathsf{MoV}_S(x,T)$ is positive if $x \in S(T)$, and negative otherwise.[6] In the unweighted setting, all MoV values are (positive or negative) integers.

Example 2.2. Consider the tournament T in Figure 1. It can be verified that $UC(T) = \{c,d,e,f\}$. For the unweighted setting, Table 2 gives the MoV values for this tournament with respect to the uncovered set, together with examples of minimum destructive or constructive reversal sets.

Note that minimum reversal sets are generally not unique, and that a minimum reversal set for an alternative x may exclusively consist of edges *not* incident to x (e.g., $\{(f,e)\}$ is a minimum CRS for b in Example 2.2).

3 Computing the Margin of Victory

In this section, we study the complexity of computing the MoV for both winners and non-winners.

3.1 Margin of Victory for Winners

For winners, we are given a tournament T, a weight function $w : E(T) \to \mathbb{R}^+$, a tournament solution S, and an alternative $x \in S(T)$; the task is to compute $\mathsf{MoV}_S(x,T)$. Clearly, a polynomial-time algorithm for the weighted setting also applies to the unweighted setting, while a hardness result in the unweighted setting implies one for the weighted setting. We remark that in all cases where we provide a polynomial-time algorithm (i.e., table entries "P" in Table 1), our algorithm not only determines the MoV value, but also finds a minimum DRS (or CRS when considering non-winners in Section 3.2).

3.1.1 Copeland

The MoV for Copeland has already been studied (under different names) in slightly different settings [Faliszewski et al., 2009, Russell and Walsh, 2009]. In particular, Theorem 3.7 of Faliszewski et al. [2009] implies that the MoV for Copeland winners can be computed efficiently whenever the weights correspond to pairwise majority margins resulting from a preference profile. For completeness, we provide a (simpler) proof tailored to our setting.[7]

Theorem 3.1. *Computing the* MoV *of a CO winner in the weighted setting can be done in polynomial time.*

Proof. Let x be the CO winner for which we want to compute the MoV. Consider a fixed minimum destructive reversal set R with T being the tournament before and T^R the tournament after the edge reversals, and let y be an alternative with higher outdegree than x in T^R. We claim that R contains

[6]The only exception is the degenerate case where S selects all alternatives for all tournaments of some size n; in this case we define $\mathsf{MoV}_S(x,T) = \infty$ for all alternatives x and all tournaments T of that size. For ease of exposition, we will assume for the rest of the paper that the degenerate case does not occur, but all of our results still hold even when this case occurs.

[7]For the *unweighted* case, a greedy approach suffices to compute the MoV of a Copeland winner. This case is not particularly interesting, however, as it can be easily verified that $\mathsf{MoV}_{CO}(x,T) = 1$ for all $x \in CO(T)$ whenever $|CO(T)| > 1$.

outgoing edges of x and ingoing edges of y only. Assume for contradiction that an edge that is neither outgoing of x nor ingoing to y is included in R. Then, deleting this edge from R does not increase the outdegree of x or decrease the outdegree of y in T^R, a contradiction to the minimality of R.

The above observation directly implies a simple polynomial-time procedure to compute a minimum destructive reversal set: Iterate over all $y \in V(T) \setminus \{x\}$ and compute the cost of a minimum reversal set that makes the outdegree of y higher than x. Up to the choice of the edge (x, y), which we handle by a case distinction, we can do so by greedily choosing outgoing edges of x and ingoing edges of y of lowest cost until y has higher outdegree than x. Among all choices of y, we select one inducing minimum cost.

To see the correctness of this algorithm, note that, after we fixed y and decided that $(x, y) \notin R$, reversing an edge outgoing of x or ingoing to y reduces the difference $|D(x)| - |D(y)|$ by 1, and y has higher outdegree than x exactly when this difference becomes negative. The same argument holds for the case that $(x, y) \in R$. □

3.1.2 Uncovered Set, *k*-Kings and Top Cycle

The problems of computing the MoV for UC, k-kings, and TC are not only closely related to each other but also to the theory of network flows. Since UC can be interpreted as 2-kings and TC as $(n-1)$-kings, we only refer to k-kings and assume that k can be chosen from $\{2, \ldots, n-1\}$. A DRS for x is then an edge set R such that x has distance greater than k to at least one alternative y in T^R.

Finding a minimum DRS is closely related to finding ℓ-length bounded s-t-cuts of minimum capacity. In the latter problem, we are given a directed network $G = (V, E)$ with a capacity function $u : E \to \mathbb{R}^+$, two distinguished nodes $s, t \in V$, and a length bound $\ell \in \mathbb{N}$. An edge set $C \subseteq E$ is an *ℓ-length bounded s-t-cut* if all s-t-paths in $(V, E \setminus C)$ have length greater than ℓ. The set C is a *minimum ℓ-length bounded s-t-cut* if it minimizes the sum of the capacities of edges in C. When $\ell \geq |V(G)| - 1$, the problem is equivalent to the standard minimum cut problem and can be solved via linear programming due to the well known max-flow min-cut theorem [Ford and Fulkerson, 1956]. However, for general $\ell \in \mathbb{N}$, Adámek and Koubek [1971] showed that a generalization of this theorem does not hold. More recently, Baier et al. [2010] proved that finding a minimum ℓ-length bounded s-t-cut is NP-hard for $\ell \in \{4, \ldots, n^{1-\epsilon}\}$ for fixed $\epsilon > 0$, even if capacities are uniform. By contrast, for $\ell \leq 3$, Mahjoub and McCormick [2010] presented a polynomial-time algorithm which reduces the problem to a standard cut problem. In the following, we show how we can extend and apply these results to our setting.

Despite its similarity to our problem (which can be observed by setting $G = T$, $u(e) = w(e)$, $\ell = k$, and $s = x$), the problem described above differs from our problem in three ways. First, the node to be disconnected from, in this case t, is specified; second, edges are removed instead of reversed; and third, the graph is not restricted to be a tournament. For ease of presentation, we define a new problem which lies between MoV for k-kings and minimum ℓ-length bounded s-t-cuts.

For a network $G = (V, E)$, we say that $C \subseteq E$ is an *ℓ-length bounded s-cut* if it is an ℓ-length bounded s-t-cut for some $t \in V \setminus \{s\}$. We say that C is a *minimum ℓ-length bounded s-cut* if it is a minimum ℓ-length bounded s-t-cut and minimizes the capacity among all $t \in V \setminus \{s\}$. Computing a minimum ℓ-length bounded s-cut can be reduced to computing a minimum ℓ-length bounded s-t-cut by iterating over all $t \in V \setminus \{s\}$. For brevity, we sometimes refer to ℓ-length bounded s-cuts and ℓ-length bounded s-t-cuts as ℓ-bounded s-cuts and ℓ-bounded s-t-cuts, respectively.

The following lemma formalizes the connection between length bounded cuts and DRSs for k-kings. We define $\mathcal{P}_{x,y}(k)$ to be the set of x-y-paths in tournament T of length at most k.

Lemma 3.2. *For each $k \in \{2, \ldots, n-1\}$, a set $R \subseteq E(T)$ is a minimum DRS for x with respect to k-kings if and only if R is a minimum k-length bounded x-cut in T.*

Proof. Fix $k \in \{2, \ldots, n-1\}$. The proof is divided into two parts. First, we show that every destructive reversal set for k-kings is also a k-bounded x-cut of equal cost, and second, we prove that every minimum k-bounded x-cut is also a destructive reversal set of equal cost. This suffices to prove the claim.

Let $R \subseteq E(T)$ be a DRS for x. Assume for contradiction that R is not a k-bounded x-cut. Recall that we assume x to be a k-king in T. Hence, for every y there exists a path $P_y \in \mathcal{P}_{x,y}(k)$ such that $P_y \cap R = \emptyset$ and therefore x can reach every y in the tournament T^R in at most k steps, a contradiction to the assumption that R is a DRS. We conclude that R is a k-bounded x-cut.

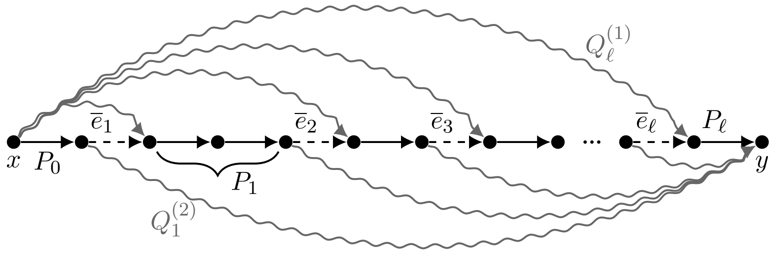

Figure 2: Illustration of a situation in which Lemma 3.2 would be violated. The path P that occurs after the reversal of edges in R is illustrated by straight edges, where edges that were reversed are dashed. Paths Q_i containing exactly one reversed edge, namely e_i, are depicted by bended, curled arrows.

For the second part we show a slightly stronger statement, namely that for every y, every minimum k-bounded x-y-cut is also a DRS of equal cost. Clearly it follows that every minimum k-bounded x-cut is a DRS of equal cost. Let $R \subseteq E(T)$ be a minimum k-bounded x-y-cut for some $y \in V(T) \setminus \{x\}$. We will show that all paths from x to y in T^R have length strictly greater than k, which suffices to prove the claim. Assume for contradiction that there exists an x-y-path P in T^R with $|P| \leq k$. Recall that $\overline{R}$ is the reversed counterpart of R and note that $P \cap \overline{R} \neq \emptyset$, since otherwise this contradicts our choice of R and y. Let $\{\overline{e}_1, \dots, \overline{e}_\ell\} := P \cap \overline{R}$ such that $\overline{e}_i$ appears before $\overline{e}_j$ in P if and only if $i < j$. We label the connected components of $P \setminus \overline{R}$ such that P_0 is the subpath of P from x to the tail of edge $\overline{e}_1$, P_ℓ is the subpath from the head of $\overline{e}_\ell$ to y, and for $i \in \{1, \dots, \ell - 1\}$, P_i is the subpath starting at the head of $\overline{e}_i$ and ending at the tail of $\overline{e}_{i+1}$. Note in particular that P_i can be empty. For every $\overline{e}_i$, recall that e_i is its non-reversed counterpart from the original tournament T. We define $\mathcal{Q}_i$ to be the set of x-y-paths in T which contain edge e_i and are not longer than k. We claim that for every e_i ($i \in \{1, \dots, \ell\}$), there exists at least one path $Q_i \in \mathcal{Q}_i$ with $Q_i \cap R = \{e_i\}$. Indeed, if this is not the case, then since $w(e_i) > 0$, the set $R \setminus \{e_i\}$ would be a feasible k-bounded x-y-cut of smaller cost, a contradiction to the minimality of R. See Figure 2 for an illustration of the situation.

From the existence of Q_i ($i \in \{1, \dots, \ell\}$), we derive the existence of x-y-paths which appear in both T and T^R and therefore need to be longer than k. To this end, we define $Q_i^{(1)}$ to be the first part of Q_i from x to the tail of edge e_i, and denote by $Q_i^{(2)}$ the second part of Q_i, i.e., the part from the head of e_i to y.

It is easy to see that $P_0 \dot\cup Q_1^{(2)}$, $Q_\ell^{(1)} \dot\cup P_\ell$ as well as $Q_i^{(1)} \dot\cup P_i \dot\cup Q_{i+1}^{(2)}$ for each $i \in \{1, \dots, \ell - 1\}$ are edge progressions from x to y, i.e., sequences of edges leading from x to y which might use nodes and edges multiple times. In particular, these multisets contain x-y-paths which do not intersect with R and therefore exist in both T and T^R. We obtain

$$|P_0| + |Q_1^{(2)}| > k \tag{1}$$

$$|Q_\ell^{(1)}| + |P_\ell| > k \tag{2}$$

$$|Q_i^{(1)}| + |P_i| + |Q_{i+1}^{(2)}| > k \qquad \forall\, i \in \{1, \dots, \ell - 1\}. \tag{3}$$

On the other hand, we know that $|P| \leq k$ and $|Q_i| \leq k$ for all $i \in \{1, \ldots, \ell\}$, which means that

$$\sum_{i=0}^{\ell} |P_i| + \ell \leq k \tag{4}$$

$$|Q_i^{(1)}| + 1 + |Q_i^{(2)}| \leq k \qquad\qquad \forall\, i \in \{1, \ldots, \ell\}. \tag{5}$$

Summing up inequalities (1)–(3) and inequalities (4)–(5) results in

$$\begin{aligned} k(\ell+1) &\overset{(1)-(3)}{<} \sum_{i=0}^{\ell} |P_i| + \sum_{i=1}^{\ell} (|Q_i^{(1)}| + |Q_i^{(2)}|) \\ &< \sum_{i=0}^{\ell} |P_i| + \sum_{i=1}^{\ell} (|Q_i^{(1)}| + |Q_i^{(2)}|) + 2\ell \\ &\overset{(4),(5)}{\leq} k(\ell+1), \end{aligned}$$

a contradiction. □

We remark that the implication from right to left in Lemma 3.2 holds only for minimum k-bounded x-cuts and not for general k-bounded x-cuts. In Figure 3 we give an example where a k-bounded x-cut does not correspond to a DRS for k-kings. In the illustrated unweighted tournament, x is a 3-king and the edge set $C = \{b, e, c\}$ is a 3-bounded x-y-cut and, hence, a 3-bounded x-cut. However, C is not a DRS as its reversal creates a new x-y-path of length three (namely, $(a, \overline{e}, d)$) and all other nodes are also still reachable in three steps.

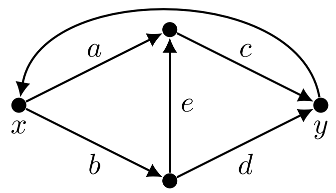

Figure 3: Example showing that the implication of Lemma 3.2 does not hold for general k-length bounded x-cuts. The edge set $\{b, e, c\}$ is a 3-length bounded x-cut, but not a DRS for x with respect to 3-kings.

Since there exist polynomial-time algorithms for computing minimum ℓ-length bounded s-t-cuts in general networks for $\ell \leq 3$ and $\ell = n-1$ [Mahjoub and McCormick, 2010, Ford and Fulkerson, 1956], Lemma 3.2 immediately yields polynomial-time algorithms for the minimum ℓ-length bounded s-cut problem in our tournament setting for $\ell \in \{2, 3, n-1\}$.

Corollary 3.3. *Computing the* MoV *of a UC winner, a* 3*-king, or a TC winner in the weighted setting can be done in polynomial time.*

Next, we show that the tractability turns into an intractability as we move from $k = 3$ to $k \geq 4$, in both the unweighted and weighted settings. Our hardness result is obtained by carefully adjusting the proof of Baier et al. [2010] showing that approximating minimum ℓ-length bounded cuts for $\ell \geq 4$ in general networks is NP-hard; we summarize our adjustments after the proof sketch. The full proof is rather involved and therefore deferred to Appendix A.

Theorem 3.4. *For any constant $k \geq 4$, computing the* MoV *of a k-king in the unweighted setting is NP-hard. For any constant $\epsilon > 0$, the problem is still NP-hard when we restrict to non-constant $k \geq n^{1-\epsilon}$.*

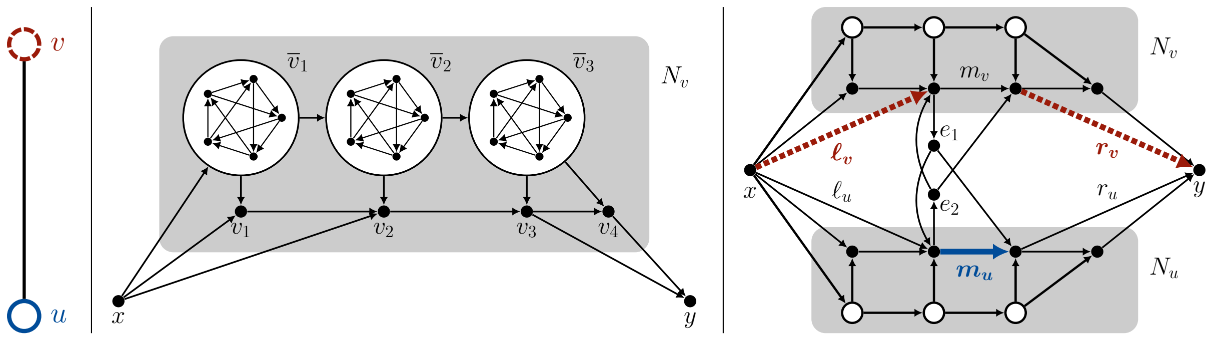

Figure 4: Illustration of the construction used in the proof of Theorem 3.4 for the case $k = 4$. For any undirected graph G (left image), a tournament T is constructed by introducing node gadgets and edge gadgets as follows. A *node gadget* N_v consists of four nodes v_1, v_2, v_3, v_4 and three "supernodes" $\overline{v}_1, \overline{v}_2, \overline{v_3}$, where the latter are tournaments themselves. The center image shows the node gadget for node v. An *edge gadget* for $e = \{u, v\}$ consists of two nodes e_1, e_2 and edges connecting the node gadgets of u and v; see the right image. Nodes x and y are connected to all node gadgets as illustrated. All omitted edges point "backwards" (from right to left) and the direction of vertical edges, if not specified, can be chosen arbitrarily.

Proof sketch. At a high level, we reduce from vertex cover; see Figure 4 for the construction for $k = 4$. Lemma 3.2 implies that determining the MoV of node x with respect to 4-kings is equivalent to computing the cost of a 4-bounded minimum x-cut. The key part of the proof is to show that, for any $c \leq |V(G)|$, there exists a vertex cover in G of size c if and only if there exists a 4-bounded x-cut in T of size $c + |V(G)|$. For the direction from left to right, a vertex cover U can be translated to a 4-bounded x-y-cut by including edges ℓ_v and r_v (depicted by red dashed edges) whenever $v \in U$ (depicted by a red dashed node) and m_v otherwise. For the other direction, we argue that any 4-bounded x-cut of size $c + |V(G)|$ can be translated to a 4-bounded x-y-cut which includes only edges of the type ℓ_v, r_v and m_v and has smaller or equal size. Reversing the previously described transformation gives us a vertex cover of size c. The proof is then extended to $k > 4$. □

Our proof of Theorem 3.4 is strongly based on the proof of Baier et al. [2010] that computing the size of minimum ℓ-length bounded s-t-cuts is NP-hard for $\ell \in \{4, \ldots, \lfloor n^{(1-\epsilon)} \rfloor\}$, where n is the size of the length bounded cut instance constructed in the reduction and $\epsilon > 0$ can be arbitrarily small. In the following we discuss the main adjustments required for the proof to work in our setting.

First, note that the minimum ℓ-length bounded s-t-cut problem does not require the graph to be a tournament, which is why we needed to alter the construction by introducing backwards and vertical edges. This increased the number of paths significantly.

Second, the problem discussed by Baier et al. [2010] specifies two nodes s and t which ought to be separated, while our problem specifies only one node x which should be separated from some node in $V(T) \setminus \{x\}$. To address this difference, we introduced supernodes to help guarantee that all x-z-cuts for $z \in V(T) \setminus \{x, y\}$ are significantly more expensive than a minimum x-y-cut. In this way, we ensure that the node y needs to be separated from x. The introduction of supernodes in turn leads to a different extension of the node gadget for $k > 4$.

3.1.3 Banks Set

Deciding whether an alternative x is contained in the Banks set of a tournament T, and hence deciding whether $\mathsf{MoV}_{BA}(x, T) > 0$, is NP-complete [Woeginger, 2003]. Our next result shows that determining $\mathsf{MoV}_{BA}(x, T)$ is computationally intractable *even* if we know that x is a Banks winner in tournament T.

Theorem 3.5. *Computing the* MoV *of a BA winner in the unweighted setting is NP-hard.*

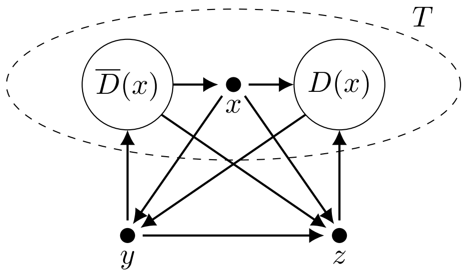

Figure 5: Illustration of the tournament T' constructed in the proof of Theorem 3.5.

Proof. We reduce from the NP-hard problem of determining whether an alternative is contained in the Banks set [Woeginger, 2003]. Take any instance of that problem, which consists of a tournament T and one of its alternatives x. Add two alternatives y, z so that y dominates only $\overline{D}(x) \cup \{z\}$, and z dominates only $D(x)$. Call the resulting tournament T' (see Figure 5). Observe that $x \in BA(T')$: the transitive subtournament $T|_{\{x,y\}}$ cannot be extended, since no alternative dominates both x and y. We claim that $\mathsf{MoV}_{BA}(x, T') = 1$ if and only if $x \notin BA(T)$.

First, assume that $x \notin BA(T)$. We show that $R = \{(x, y)\}$ is a DRS for x. Consider any transitive subtournament in $T'' = (T')^R$ with x as the maximal element. This tournament cannot include y, but may include z. Since $x \notin BA(T)$, there exists an alternative w in T that dominates all alternatives in the subtournament. In particular, since $w \in \overline{D}(x)$, w also dominates z. Hence the transitive subtournament can be extended by w, implying that $x \notin BA(T'')$.

Assume now that $x \in BA(T)$. We claim that $\mathsf{MoV}_{BA}(x, T') > 1$. Since $x \in BA(T)$, there exists a transitive subtournament in T with x as the maximal element that cannot be extended by any alternative in T. Moreover, since x dominates both y and z, this subtournament cannot be extended by y or z. Unless we reverse an edge in T or the edge (x, z), this subtournament still cannot be extended. If we reverse the edge (x, z), the transitive subtournament $T|_{\{x,y\}}$ cannot be extended. Else, if we reverse an edge in T, the transitive subtournament $T|_{\{x,y,z\}}$ cannot be extended. Hence, there is no DRS for x of size one, as claimed. $\square$

3.2 Margin of Victory for Non-Winners

In this subsection, we consider the problem of computing the MoV for non-winners.

3.2.1 Copeland

Similarly to the winner case, the results of Faliszewski et al. [2009] already imply that the MoV for non-winners can be computed in polynomial time. For completeness, we remark that a greedy algorithm suffices for our unweighted setting, and present a network flow approach for the weighted case.

Theorem 3.6. *Computing the* MoV *of a CO non-winner in the weighted setting can be done in polynomial time.*

Proof. We aim to compute the MoV of a Copeland non-winner x. Any member of the Copeland set needs to have an outdegree of at least $\lceil (n-1)/2 \rceil$. We iterate over all $c \in \{\lceil (n-1)/2 \rceil, \dots, n\}$ and compute the minimum cost of making x a Copeland winner given that the outdegree of all Copeland winners is c after the reversals. To this end, we construct a network G where $V(G) = E(T) \cup V(T) \cup \{s, t\}$, with $E(T)$ being the edge set of the tournament T. We will specify the edges of G later.

We consider a slightly non-standard definition of a network in which edges have associated costs as well as capacities and nodes have balances. This definition allows us to search for "b-flows" of minimum cost and is discussed in detail, e.g., by Korte and Vygen [2012]. Informally speaking, the balance of

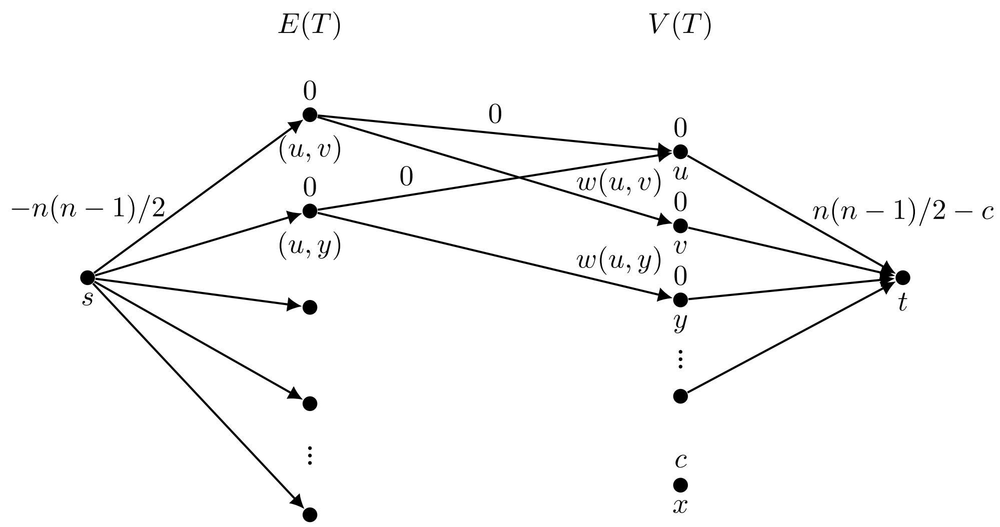

Figure 6: Illustration of the construction used in the proof of Theorem 3.6. Each node has two labels, i.e., its name below and its balance above the corresponding circle. The cost of an edge is notated above its associated arrow and capacities of edges are omitted.

a node corresponds to the amount of flow this node absorbs (or rather produces, in case of a negative value) and a b-flow is a flow which respects the induced constraints. We define the balances in our network construction as follows:

$$b_v = \begin{cases} -n(n-1)/2 & \text{if } v = s; \\ c & \text{if } v = x; \\ n(n-1)/2 - c & \text{if } v = t; \\ 0 & \text{else.} \end{cases}$$

There exists an edge from $e \in E(T)$ to $v \in V(T)$ if and only if v is one of the endnodes of the edge e in the tournament graph T. The edge (e, v) has cost 0 if v is the tail of edge e; otherwise its cost is equal to the cost of reversing e in the tournament graph. All of these edges have capacity 1. In addition, there exist edges from s to each node in $E(T)$ with zero cost and capacity 1, as well as edges from each node in $V(T) \setminus \{x\}$ to t with zero cost and capacity c.

We claim that from an integral b-flow in G with capacity threshold c, we can construct a constructive reversal set for x of equal cost such that x is a Copeland winner with outdegree c in T^R, and vice versa. Consider the illustration of the construction in Figure 6, in which the nodes are arranged in four levels, where the first level contains the source s, the second-level nodes correspond to the edges in T, the third-level nodes correspond to the nodes in T, and the last level contains the sink t. The nodes in the second layer have two outgoing edges in the network, representing the choices between *keeping the direction of the corresponding edge in T as it currently is* and *reversing the edge* in the tournament. Any feasible integral flow can only send flow along one of the edges.

More precisely, given an integral b-flow, the reversal set R is determined by the edges pointing from the second to the third layer with non-zero cost and flow value 1. The amount of flow that reaches a node in the third level corresponds exactly to the outdegree of this node in the tournament T^R. Since edges from the third to the fourth level have capacity c, a feasible flow guarantees that any node in $V(T)$ has outdegree at most c. Moreover, the node x has a balance of c and therefore it has outdegree exactly c in T^R. Hence, R is a constructive reversal set with cost equal to the cost of the flow. The other direction follows due to similar arguments.

Integral b-flows of minimum cost can be found in polynomial time, for example by the minimum mean cycle-cancelling algorithm [Klein, 1967, Goldberg and Tarjan, 1989].[8] After repeating the construction for all $c \in \{\lceil (n-1) \rceil /2, \ldots, n\}$, we choose a constructive reversal set with minimum cost. $\square$

3.2.2 Uncovered Set, k-Kings, and Top Cycle

To get a non-winner x into the uncovered set, we need its dominion to be a dominating set in T_{-x}. Since a tournament with n vertices always has a dominating set of size $\lceil \log_2 n \rceil$ [Megiddo and Vishkin, 1988], we do not need to reverse more than $\lceil \log_2 n \rceil$ edges. This also means that there exists an $n^{O(\log n)}$ algorithm for finding the minimum number of necessary edge reversals, as we can try all combinations of at most $\lceil \log_2 n \rceil$ vertices to add to the dominion of x. Megiddo and Vishkin [1988] also proved that the problem of finding a dominating set of minimum size in a tournament, which we call MINIMUM DOMINATING SET, is unlikely to admit a polynomial-time algorithm: the existence of such an algorithm would have unexpected implications on the satisfiability problem[9] We present a reduction from MINIMUM DOMINATING SET to the problem of computing the MoV for UC non-winners, which means that the latter problem is also unlikely to admit an efficient algorithm. Our reduction is similar to the one used by Yang and Guo [2017, Theorem 3] to show a parameterized complexity result for the decision version of the problem.

Theorem 3.7. *Computing the* MoV *of a UC non-winner in the unweighted setting is at least as hard as* MINIMUM DOMINATING SET.

Proof. Consider an instance of MINIMUM DOMINATING SET given by a tournament T. Define a new tournament T' by adding an alternative $x \notin V(T)$ to T, and by making x a Condorcet loser in T'. We claim that $-\mathsf{MoV}_{UC}(x, T')$ is equal to the minimum size of a dominating set in T. For any dominating set in T, we obtain a constructive reversal set for x in T' consisting of the edges between x and all members of the set. On the other hand, consider a CRS R for x in T'. Suppose that R contains an edge (z, y) with $x \notin \{z, y\}$, such that $y \succ z$ in T'^R. The only alternative that this reversal can help x reach in two steps is z. In this case, we can instead include (z, x) in R and maintain the property that x can reach all other alternatives in at most two steps. Hence there is always a minimum CRS that only contains edges incident to x. The alternatives involved in this CRS besides x form a dominating set in T. $\square$

In the unweighted setting, minimum CRSs with respect to k-kings ($k \geq 3$) are single edges (see Theorem 4.2) and hence can be found efficiently. In the weighted setting, we show hardness for UC and k-kings and tractability for TC.

Theorem 3.8. *Computing the* MoV *of a UC non-winner in the weighted setting is NP-hard.*

Proof. Given an instance of SET COVER with a universe of size r and a collection of s sets, we construct a tournament with alternatives x, y and alternative sets A, B, where $|A| = s$ and $|B| = r$. The edges are given as follows (see also Figure 7):

- $A \succ x \succ y \succ A$;
- $B \succ \{x, y\}$;
- Each alternative in A dominates the corresponding subset of B in the SET COVER instance, and is dominated by the remaining alternatives in B.

[8] The minimum mean cycle-cancelling algorithm computes integral b-flows if b and the capacities in the network are integers, which is the case in our setting.

[9] Papadimitriou and Yannakakis [1996] proved that MINIMUM DOMINATING SET is complete with respect to the class LOGSNP, which lies between P and NP. Moreover, although the problem is unlikely to be NP-hard, Downey and Fellows [1995] showed that it is W[2]-hard with respect to the number of reversed edges.

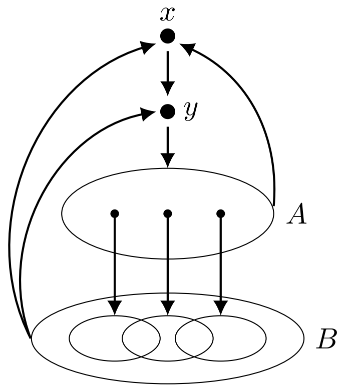

Figure 7: Illustration of the construction used in the proof of Theorem 3.8.

The edges within A and B are arbitrary. The edges between A and x have cost 1, while the remaining edges have cost n^2, where n denotes the number of vertices in the constructed tournament.

The chosen costs imply that a miminum CRS will only contain edges between A and x. Since x already reaches all alternatives of A in two steps via y, it only needs to reach all vertices of B in two steps via A in order to be a UC winner. Therefore, the minimum cost of a CRS is exactly the size of a minimum set cover. $\square$

Theorem 3.9. *For any constant $k \geq 3$, computing the* MoV *of a non-k-king in the weighted setting is NP-hard. For any constant $\epsilon > 0$, the problem is still NP-hard when we restrict to non-constant $k \geq (1 - \epsilon)n$.*

Proof. The proof is divided into two parts. First, we introduce the reduction for any constant $k \geq 3$. Second, we argue that, for any $\epsilon > 0$, we can still carry out the reduction even when we restrict ourselves to the problem in which $k \geq (1 - \epsilon)n$.

We use a similar reduction as for UC. Instead of having a single alternative y, we add $k - 1$ alternatives $y_1, \ldots, y_{k-1}$ such that

- $x \succ y_1 \succ \cdots \succ y_{k-1} \succ A$;
- $y_i \succ x$ for $i \geq 2$;
- $y_j \succ y_i$ for $j \geq i + 2$;
- $A \succ \{x, y_1, \ldots, y_{k-2}\}$;
- $B \succ \{x, y_1, \ldots, y_{k-2}, y_{k-1}\}$;
- Each alternative in A dominates the corresponding subset of B in the SET COVER instance, and is dominated by the remaining alternatives in B.

The edges within A and B are arbitrary. The edges between A and y_{k-2} have cost 1, while the remaining edges have cost n^2.

The choice of edge costs implies that a minimum CRS only contains edges between A and y_{k-2}. Since x already reaches all alternatives of A in k steps via $y_1, \ldots, y_{k-1}$, it only needs to reach all vertices of B in k steps via $y_1, \ldots, y_{k-2}, A$ in order to be part of the uncovered set. Therefore, the minimum cost of a CRS is exactly the size of a minimum set cover.

It remains to argue that even if we restrict ourselves to the problem with $k \geq (1 - \epsilon)n$, for any fixed $\epsilon > 0$, we can still carry out the above reduction in polynomial time. Let $\epsilon > 0$ be given and let r be the size of the universe and s be the number of sets in the SET COVER instance. We define n_k to be the number of nodes in the tournament which we construct for a given k. More precisely, it holds that

$$n_k = k + s + r.$$

Choose the smallest $k \in \mathbb{N}_{\geq 3}$ such that

$$k \geq (s + r) \left(\frac{1 - \epsilon}{\epsilon} \right).$$

This is still polynomial in s and r, and implies that

$$k = (1 - \epsilon) \left(k + \frac{\epsilon}{1 - \epsilon} k \right) \geq (1 - \epsilon)(k + s + r) = (1 - \epsilon) n_k,$$

concluding the proof. □

Theorem 3.10. *Computing the* MoV *of a TC non-winner in the weighted setting can be done in polynomial time.*

Proof. Consider a partition of T into strongly connected components. These components form a linear order with all vertices in an earlier component dominating all vertices in a later component. Call the components $T_1, \dots, T_k$ according to the linear order, and assume that x belongs to T_r. Since $x \notin TC(T) = V(T_1)$, we have $r \geq 2$. Construct a tournament T' with vertices $v_1, \dots, v_r$. For $1 \leq i < j \leq r - 1$, add a directed edge $v_j \succ v_i$ with cost equal to the minimum cost of an edge between an alternative in T_i and an alternative in T_j. For $1 \leq i \leq r - 1$, add a directed edge $v_r \succ v_i$ with cost equal to the minimum cost of an edge between an alternative in $T_r \cup T_{r+1} \cup \dots \cup T_k$ and an alternative in T_i.

We claim that the shortest path distance from v_r to v_1 in T' equals the minimum cost of a CRS for x in T. Take any shortest path from v_r to v_1. For each edge on this path, we reverse a corresponding edge in T with the same cost. This allows x to reach all components corresponding to vertices on this path, including T_1. Note that x can already reach the components $T_{r+1}, \dots, T_k$ even before the reversals. Moreover, the remaining components are directly reachable from T_1, and therefore x can also reach them.[10] Hence we can bring x into TC using no more cost than that of the shortest path. On the other hand, any CRS for x must have the effect that x can reach an alternative in T_1. This gives rise to a path from v_r to v_1 in T' with no greater cost.

Computing strongly connected components of T can be done in time linear in the input size using Tarjan's algorithm or Kosaraju's algorithm, and finding the shortest path can be done in polynomial time using Dijkstra's algorithm. Therefore our algorithm runs in polynomial time. □

3.2.3 Banks Set

For Banks non-winners, we present an analogous result as in the winner case: even if we know that x has a negative MoV in tournament T, determining $\text{MoV}_{BA}(x, T)$ is intractable.

Theorem 3.11. *Computing the* MoV *of a BA non-winner in the unweighted setting is NP-hard.*

Proof. We reduce from the NP-hard problem of determining whether an alternative is contained in the Banks set [Woeginger, 2003]. Take any instance of that problem, which consists of a tournament T and one of its alternative x. Add two alternatives y, z so that y dominates $D(x) \cup \{x, z\}$ but is dominated by $\overline{D}(x)$, while z dominates all alternatives in T. Call the resulting tournament T' (see Figure 8). Observe

[10] An exception to this is if we reverse an edge $v_p \succ v_1$, and both T_p and T_1 are singletons. However, in this case x can reach T_p via the shortest path.

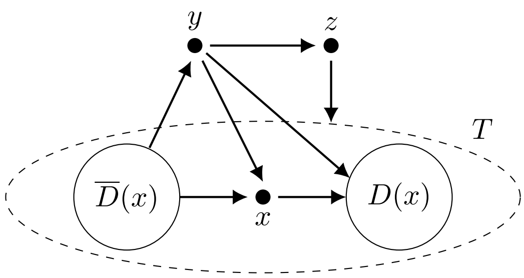

Figure 8: Illustration of the construction used in the proof of Theorem 3.11.

that $x \notin BA(T')$: any transitive subtournament with x as the maximal element cannot contain y, and can therefore be extended by z. Hence, $\mathsf{MoV}_{BA}(x, T') < 0$. We claim that $\mathsf{MoV}_{BA}(x, T') = -1$ if and only if $x \in BA(T)$.

First, assume that $x \in BA(T)$. This means x is the maximal element of a transitive subtournament T'' of T that cannot be extended by any alternative in T. Reverse the edge (x, y), and insert y into T'' at the position after x. The resulting subtournament cannot be extended by any alternative in T, nor can it be extended by z because $y \succ z$. Hence $x \in BA(T')$ after the reversal.

Assume now that $x \notin BA(T)$, and suppose for contradiction that $\mathsf{MoV}_{BA}(x, T') = -1$. Notice that x is covered by both y and z, so in order to get x into $BA(T')$, we need to either strengthen x, or weaken both y and z. The latter option cannot be accomplished with one reversal, so the reversal needs to strengthen x. If it strengthens x against another alternative in T, x is still covered by z. If it strengthens x against z, x is still covered by y. Hence the reversal must strengthen x against y. Consider any transitive subtournament T'' after the reversal with x as the maximal element. The tournament T'' cannot contain z, but may contain y. However, since y has the same dominion as x in T, an alternative that extends T''_{-y}, which must exist because $x \notin BA(T)$, necessarily extends T''. This yields the desired contradiction. □

4 Bounds on the Margin of Victory

For the rest of the paper, we will focus on the unweighted setting. In this section, we establish bounds on the MoV values for both winners and non-winners. Before we proceed to the bounds, we remark that there are at least two insights that one could draw from these bounds. First, tournament solutions with a low absolute value of MoV bound yield manipulability guarantees; indeed, if the absolute value of the MoV bound is low, then a manipulator can always obtain the desired outcome by reversing a small number of edges regardless of the tournament instance. Second, knowing these bounds is useful for understanding the actual MoV for specific tournaments. For example, one can calculate the "relative/normalized MoV" by dividing the actual MoV value by the bound. The resulting ratio provides a measure of how far away an alternative is from winning or losing; in contrast to the standard MoV measure, the relative MoV also enables us to make comparisons between tournaments of different sizes.

4.1 Winners

We show that for all considered tournament solutions, one may need to reverse up to $\lfloor n/2 \rfloor$ edges to take a winner out of the winner set, but no more.

Theorem 4.1. *Let $S \in \{CO, TC, UC, BA, k\text{-kings}\}$, where $k \geq 3$. For any tournament T and any $x \in S(T)$, we have $\mathsf{MoV}_S(x, T) \leq \lfloor n/2 \rfloor$. Moreover, this bound is tight.*

Proof. Since all of the tournament solutions considered are contained in TC, an upper bound for TC

carries over to the other solutions as well. By analogous reasoning, it suffices to show the tightness of the bound for BA and CO.

We first prove the upper bound. Let y be an arbitrary Copeland winner in T_{-x}. Since T_{-x} consists of $n-1$ alternatives, y dominates at least $\lceil (n-2)/2 \rceil = \lceil n/2 \rceil - 1$ other alternatives. Hence, we can make y a Condorcet winner in T by reversing at most $(n-1) - (\lceil n/2 \rceil - 1) = \lfloor n/2 \rfloor$ edges. Since TC is Condorcet-consistent, $\lfloor n/2 \rfloor$ edge reversals suffice to take x out of TC.

Next, we show the lower bound for BA. Assume first that n is even, say $n = 2\ell$. Besides x, suppose that T contains alternatives $y_1, \dots, y_{2\ell-1}$, which are placed around a circle in clockwise order. Each alternative dominates the $\ell - 1$ following alternatives in clockwise order (e.g., y_1 dominates $y_2, \dots, y_\ell$), and all $2\ell - 1$ alternatives are dominated by x. We claim that taking x out of the Banks set requires at least $\lfloor n/2 \rfloor = \ell$ edge reversals. Consider the $2\ell - 1$ sets

$$\{y_1, y_\ell\}, \{y_2, y_{\ell+1}\}, \dots, \{y_\ell, y_{2\ell-1}\},$$
$$\{y_{\ell+1}, y_1\}, \dots, \{y_{2\ell-1}, y_{\ell-1}\}.$$

Note that each y_i is contained in exactly two of these sets. For each set, we say that it is 'good' if the only alternative that dominates both of the alternatives in the set is x, and 'bad' otherwise. Note that the existence of a good set implies that x is a Banks winner, as the transitive subtournament consisting of the good set and x cannot be extended by another alternative that dominates all alternatives in this subtournament. Initially, all $2\ell - 1$ sets are good. A reversal involving x and y_i can turn at most two good sets into bad sets (i.e., the two sets containing y_i). Similarly, a reversal involving y_i and y_j, where y_j dominates y_i after the reversal, can make at most two good sets bad (i.e., the two sets containing y_i). So after at most $\ell - 1$ reversals, at least one set is still good. This implies that there is no DRS of size at most $\ell - 1$. Hence, $\mathsf{MoV}_{BA}(x, T) \geq \ell = \lfloor n/2 \rfloor$.

The case where n is odd can be handled similarly. Let $n = 2\ell - 1$. Construct a tournament with alternatives $x, y_1, \dots, y_{2\ell-1}$ as before, and remove $y_{2\ell-1}$. We claim that taking x out of the Banks set in this tournament requires at least $\lfloor n/2 \rfloor = \ell - 1$ edge reversals. Consider $2\ell - 3$ sets, starting with the $2\ell - 1$ sets above and removing the two sets that contain $y_{2\ell-1}$. Each y_i is contained in at most two of these sets. The previous argument can be applied to show that $\mathsf{MoV}_{BA}(x, T) \geq \ell - 1 = \lfloor n/2 \rfloor$.

To conclude the proof, we show that the same tournaments as constructed above for BA also imply the tightness of the bound $\lfloor n/2 \rfloor$ for CO. In order to make x a non-winner, we must reverse edges so that another alternative y has a larger dominion than x. If n is even, then initially x dominates $n - 1$ alternatives while y dominates $n/2 - 1$ alternatives, so $|D(x)| - |D(y)| = (n-1) - (n/2 - 1) = n/2$. Each edge reversal decreases this difference by at most 1, except for the reversal of the edge (x, y), which reduces the difference by 2. Hence, in order to make the difference negative, we need at least $n/2$ reversals. A similar argument applies for the case where n is odd, since we have $|D(y)| \leq (n-1)/2$, and therefore $|D(x)| - |D(y)| \geq (n-1)/2 = \lfloor n/2 \rfloor$. $\square$

4.2 Non-Winners

Next, we turn our attention to non-winners. For TC and k-kings with $k \geq 3$, it is clear that reversing one edge suffices to make any alternative x a winner. Indeed, we can simply reverse the edge between x and an arbitrary alternative in the uncovered set of T_{-x}. This ensures that x can reach every other alternative via a directed path of length at most three.[11]

Theorem 4.2. *Let $S \in \{TC, k\text{-kings}\}$, where $k \geq 3$ is arbitrary. For any tournament T and any $x \in V(T) \setminus S(T)$, we have $\mathsf{MoV}_S(x, T) = -1$.*

For CO, as many as $n - 2$ edge reversals may be required.

[11]The statement for TC has also been shown by Yang [2017]; see the Claim in the proof of Theorem 9 in the extended version of his paper.

Theorem 4.3. *For any tournament T and any $x \in V(T) \setminus CO(T)$, we have $\mathsf{MoV}_{CO}(x,T) \geq -(n-2)$. Moreover, this bound is tight.*

Proof. With a budget of $n-2$ reversals, we can make x dominate at least $n-2$ alternatives. Moreover, if the tournament initially contains a Condorcet winner, one of these reversals can be used to make x dominate it, meaning that every alternative dominates at most $n-2$ alternatives after the reversals. Hence x becomes a Copeland winner.

To show tightness, consider a tournament where x is a Condorcet loser and there is a Condorcet winner y. We have $|D(y)| - |D(x)| = n-1$. Each edge reversal reduces this difference by at most 1, except for the reversal of the edge (x,y), which reduces the difference by 2. In order for x to be a Copeland winner, this difference must be nonpositive. It follows that we need at least $n-2$ reversals, as claimed. □

Finally, we show that for UC and BA, reversing $O(\log n)$ edges can bring any alternative into the winner set.

Theorem 4.4. *Let $S \in \{UC, BA\}$. For any tournament T and any $x \in V(T) \setminus S(T)$, we have $\mathsf{MoV}_S(x,T) \geq -\lceil \log_2 n \rceil$. Moreover, this bound is asymptotically tight.*

Proof. Since $BA \subseteq UC$, it suffices to establish the bound for BA and the tightness for UC. We first prove the bound for BA, by considering a tournament T and iteratively constructing a CRS for an alternative $x \notin BA(T)$. Let T' be a transitive subtournament of T that initially contains only the alternative x, and let B be the set of alternatives that dominate all alternatives in T'. Let $\ell = |B|$, and let y be a Copeland winner of the tournament $T|_B$. Note that y dominates at least $\lceil (\ell-1)/2 \rceil$ other alternatives in B as well as x. We reverse the edge between x and y, insert y into the transitive tournament T' at the position after x, and update the set B. Since y is added to T', y and all alternatives dominated by y are no longer in B. Also, no new alternative is added into B. Hence the size of B reduces to at most $\ell - 1 - \lceil (\ell-1)/2 \rceil = \lfloor (\ell-1)/2 \rfloor$. Since $|B| \leq n-1$ at the beginning, the size of B becomes 0 after at most $\lceil \log_2 n \rceil$ reversals, at which point $x \in BA(T)$.

To show the asymptotic tightness for UC, assume that x is a Condorcet loser and T_{-x} is a tournament for which any dominating set has size $\Omega(\log n)$; such a tournament is known to exist [Erdős, 1963, Graham and Spencer, 1971]. Let $R \subseteq E(T)$ be a CRS for x with respect to UC. Observe that if there is an edge $(y,z) \in R$ such that $x \notin \{y,z\}$, then by replacing (y,z) with (y,x) (or simply removing (y,z) if (y,x) already belongs to R), the resulting set R' is still a CRS for x. Moreover, $|R'| \leq |R|$. Therefore we may assume that all edges in R are incident to x; let these edges be $(y_1,x),\ldots,(y_{|R|},x)$. Since $x \in UC(T^R)$, the set $\{y_1,\ldots,y_{|R|}\}$ necessarily forms a dominating set in T_{-x}. It follows that $|R| \in \Omega(\log n)$, as desired. □

5 Structural Results

Our results so far have shed light on the computational properties and bounds for the MoV notion. In this section, we improve our understanding of the MoV from an axiomatic perspective by providing a number of results relating the MoV to structural properties of the tournament in question. In particular, we identify conditions on tournament solutions ensuring that the corresponding MoV values are consistent with the covering relation (Section 5.1), and examine the relationship between MoV values and Copeland scores (Sections 5.2 and 5.3). We also consider monotonicity of the MoV in Section 5.4. Some of our results are summarized in Table 3.

	cover-cons.	strong degree-cons.	degree-cons.	equal-degree-cons.
MoV_{CO}	✓	✗	✓	✗
MoV_{TC}	✓	✓	✓	✓
MoV_{UC}	✓	✗	✗	✗
$\mathsf{MoV}_{k\text{-kings}}$	✓	✗	✗	✗
MoV_{BA}	✓	✗	✗	✗

Table 3: Consistency properties of the margin of victory for the considered tournament solutions.

5.1 Cover-Consistency

Recall from Section 2 that an alternative x *covers* another alternative y if $D(y) \subseteq D(x)$. In particular, this implies that x dominates y (as otherwise $x \in D(y)$). The covering relation, which forms the basis for defining the uncovered set UC, is transitive and has a close connection to Pareto dominance in voting settings [Brandt et al., 2016b].

Intuitively, if x covers y, there is a strong argument that x is preferable to y. We show that for all of the tournament solutions that we consider, their corresponding MoV values are indeed consistent with this intuition.

Definition 5.1. For a tournament solution S, we say that MoV_S is *cover-consistent* if, for any tournament T and any alternatives $x, y \in V(T)$, x covers y implies $\mathsf{MoV}_S(x,T) \geq \mathsf{MoV}_S(y,T)$.

We introduce a new property for tournament solutions that will be useful for showing the cover-consistency of MoV functions; the property may be of independent interest in the general study of tournament solutions.

Definition 5.2. A tournament solution S is *transfer-monotonic* if for any edges $(y,z),(z,x) \in E(T)$,

$$x \in S(T) \quad \text{implies} \quad x \in S(T'),$$

where T' is the tournament obtained from T by reversing edges (y,z) and (z,x).

In other words, if an alternative x is chosen, then it remains chosen when an alternative z is "transferred" from the dominion $D(y)$ of another alternative y to the dominion $D(x)$ of x.

We show that monotonicity and transfer-monotonicity together imply cover-consistency of the margin of victory.

Lemma 5.3. *If a tournament solution S is monotonic and transfer-monotonic, then* MoV_S *satisfies cover-consistency.*

Proof. Let S be a monotonic and transfer-monotonic tournament solution, and suppose that alternative x covers another alternative y in a tournament T. We will show that $\mathsf{MoV}_S(x,T) \geq \mathsf{MoV}_S(y,T)$.

If $x \in S(T)$ and $y \notin S(T)$, the statement holds trivially since $\mathsf{MoV}_S(x,T) > 0 > \mathsf{MoV}_S(y,T)$. Suppose for contradiction that $x \notin S(T)$ and $y \in S(T)$. Consider the tournament T' obtained from T by reversing the edge (x,y) as well as edges $(x,z),(z,y)$ for each $z \in D(x) \setminus (D(y) \cup \{y\})$. By monotonicity and transfer-monotonicity, $y \in S(T')$. However, tournaments T and T' are isomorphic, and there is an isomorphism that maps $x \in T$ to $y \in T'$. Since $x \notin S(T)$, we must have $y \notin S(T')$, a contradiction.

The remaining two cases are $x, y \in S(T)$ and $x, y \notin S(T)$; both can be handled in an analogous manner, so let us focus on the latter case. It suffices to show that given any CRS for y of minimum size,

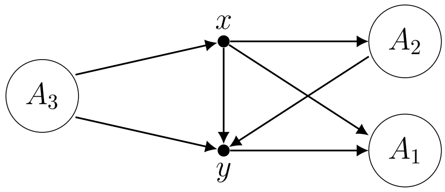

Figure 9: Illustration of the construction used in the proof of Lemma 5.3.

we can construct a CRS of smaller or equal size for x. Let R_y be a CRS for y of minimum size; we will construct a CRS R_x for x such that $|R_x| \leq |R_y|$.

Let $A = V(T) \setminus \{x, y\}$, and partition A into three sets $A_1 = D(y)$, $A_2 = D(x) \setminus (D(y) \cup \{y\})$, and $A_3 = \overline{D}(x)$; see Figure 9 for an illustration. For any edge in R_y between two alternatives of A, we add the same edge to R_x. We do not add the edge (x, y) regardless of whether it is present in R_y. Each remaining edge in R_y is between an alternative in A and one of x, y. Note that $(y, a) \notin R_y$ for any $a \in A$—otherwise, by monotonicity, removing such an edge would keep R_y a CRS for y, contradicting the minimality of R_y.

For each $a \in A$, we add further edges to R_x as follows.

- For $a \in A_1$:
  - If $(x, a) \in R_y$, add (y, a) to R_x.
- For $a \in A_2$:
  - If $(x, a) \in R_y$ but $(a, y) \notin R_y$, add (x, a) to R_x.
  - If $(x, a) \notin R_y$ but $(a, y) \in R_y$, add (a, y) to R_x.
- For $a \in A_3$:
  - If $(a, x) \in R_y$, add (a, y) to R_x.
  - If $(a, y) \in R_y$, add (a, x) to R_x.

Clearly, $|R_x| \leq |R_y|$, and we have $y \in S(T^{R_y})$ by definition of R_y. From T^{R_y}, we reverse the edge (x, y) if it is present, and for $a \in A_2$ such that both $(x, a), (a, y) \notin R_y$, we reverse (x, a) and (a, y). Let T' be the resulting tournament. By monotonicity and transfer-monotonicity, we have $y \in S(T')$. However, one can verify that there exists an isomorphism from T' to T^{R_x} that maps x to y, y to x, and every other alternative a to itself. Since $y \in S(T')$, we must have $x \in S(T^{R_x})$, meaning that R_x is indeed a CRS for x. □

In Appendix B.1, we show that neither monotonicity nor transfer-monotonicity can be dropped from the condition of Lemma 5.3. This also means that neither of the two properties implies the other.

We now show that all tournament solutions we consider in this paper satisfy both monotonicity and transfer-monotonicity, thereby implying that their MoV functions are cover-consistent.

Proposition 5.4. *CO, UC, TC, k-kings, and BA satisfy monotonicity.*

Proof. It is already known that CO, UC, TC, and BA are monotonic [Laslier, 1997, Brandt et al., 2016a]; hence, it remains to establish the monotonicity of k-kings. Let x be a k-king in tournament T, and suppose that T' is the tournament obtained by reversing an edge (y, x). Since any path of length at most k from x to another alternative in T cannot contain the edge (y, x), the same path is also present in T'. Hence x is also a k-king in T'. □

Proposition 5.5. *CO, UC, TC, k-kings, and BA satisfy transfer-monotonicity.*

Proof. We start with CO. If $x \in CO(T)$ and edges (y, z) and (z, x) are reversed, then the outdegree of x increases by 1, that of y decreases by 1, while all other alternatives have the same outdegree as before. Hence x is in the Copeland set of the new tournament.

Next, we turn to k-kings. Let x be a k-king in tournament T, and suppose that T' is the tournament obtained by reversing edges (y, z) and (z, x). Consider a path of length at most k from x to another alternative w in T; this path cannot contain the edge (z, x). If the path does not contain the edge (y, z), then the same path also exists in T'. Else, the path has the form $x \to \cdots \to y \to z \to \cdots \to w$, where possibly $z = w$. We may then shorten this path to $x \to z \to \cdots \to w$ in T', meaning that x can also reach w in at most k steps in T'. The proofs for UC and TC proceed in a similar manner.

Finally, let $x \in BA(T)$, and consider an inclusion-maximal transitive subtournament with x as the Condorcet winner. Define T' as in the previous paragraph. Since $(z, x) \in E(T)$, z does not belong to the subtournament, so all edges of the subtournament are intact in T'. Since the subtournament is inclusion-maximal in T, no alternative different from z extends it in T'. Moreover, since $(x, z) \in E(T')$, z cannot extend the subtournament in T' either. It follows that the subtournament is inclusion-maximal in T', and therefore $x \in BA(T')$. $\square$

Lemma 5.3 and Propositions 5.4 and 5.5 together imply the following:

Theorem 5.6. *For each $S \in \{CO, TC, UC, k\text{-kings}, BA\}$, MoV_S satisfies cover-consistency.*

In light of Theorem 5.6, one may wonder whether a stronger property, in which x covers y implies the strict inequality $\mathsf{MoV}_S(x) > \mathsf{MoV}_S(y)$, can also be achieved. However, the answer is negative for all Condorcet-consistent tournament solutions, including all solutions that we consider. Indeed, in a transitive tournament $x \succ y \succ z$ of size 3, such a solution only selects x. But since all three alternatives are chosen when they form a cycle (due to symmetry), both y and z can be brought into the winner set by reversing only one edge, so $\mathsf{MoV}_S(y) = -1 = \mathsf{MoV}_S(z)$ even though y covers z.

5.2 Degree-Consistency

Given a tournament solution S and a tournament T, the MoV_S values yield a natural *ranking* (possibly including ties) of the alternatives in T, where alternative x is ranked higher than y whenever $\mathsf{MoV}_S(x, T) > \mathsf{MoV}_S(y, T)$. We are interested in how closely this ranking by MoV values resembles the ranking by Copeland scores, according to which x is ranked higher than y if $\text{outdeg}(x) > \text{outdeg}(y)$.

Definition 5.7. For a tournament solution S, we say that MoV_S satisfies

- *degree-consistency* if, for any tournament T and any alternatives $x, y \in V(T)$, $\text{outdeg}(x) > \text{outdeg}(y)$ implies $\mathsf{MoV}_S(x, T) \geq \mathsf{MoV}_S(y, T)$;
- *equal-degree-consistency* if, for any tournament T and any alternatives $x, y \in V(T)$, $\text{outdeg}(x) = \text{outdeg}(y)$ implies $\mathsf{MoV}_S(x, T) = \mathsf{MoV}_S(y, T)$; and
- *strong degree-consistency* if, for any tournament T and any alternatives $x, y \in V(T)$, $\text{outdeg}(x) \geq \text{outdeg}(y)$ implies $\mathsf{MoV}_S(x, T) \geq \mathsf{MoV}_S(y, T)$.

It follows directly from the definitions that for any tournament solution S, MoV_S satisfies strong degree-consistency if and only if it satisfies both degree-consistency and equal-degree-consistency. Observe also that cover-consistency is implied by degree-consistency.

We remark that these properties are not necessarily desirable from a normative perspective: Whereas the ranking implied by a strongly degree-consistent MoV function merely represents a coarsening of the straightforward ranking by outdegree, we are often interested in tournament solutions that take more structure of the tournament into account and, as a consequence, have MoV functions that may violate (equal-)degree-consistency. Indeed, since degree-consistent MoV functions are in line with Copeland scores, their significance is somewhat limited and there would be little additional value derived from the MoV computations, which in some cases are much more involved than simply calculating Copeland scores.

5.2.1 Copeland Set

We start by showing that MoV_{CO} satisfies degree-consistency but not equal-degree-consistency.

Proposition 5.8. MoV_{CO} *does not satisfy equal-degree-consistency.*

Proof. We construct a counterexample with seven alternatives x, z, y_1, y_2, y_3, y_4 and y_5; see Figure 10 for an illustration. Alternative z is the unique Copeland winner with an outdegree of 5, alternatives x, y_3, y_4 and y_5 have outdegree 3, and y_1 and y_2 have outdegree 2. We argue that, even though y_3 and x have the same outdegree, it holds that $\mathsf{MoV}_{CO}(y_3, T) = -1$ and $\mathsf{MoV}_{CO}(x, T) = -2$. The former holds since y_3 can be made a Copeland winner by reversing the edge (y_3, z). For x, however, there does not exist an edge whose reversal simultaneously strengthens x and weakens z. Hence, we need to reverse at least two edges, e.g., (x, y_3) and (x, y_4), in order to make x a Copeland winner. $\square$

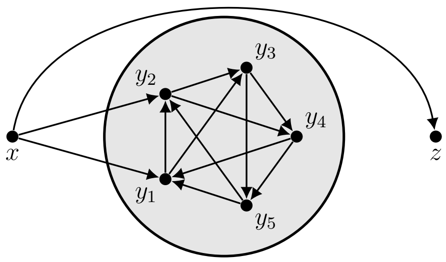

Figure 10: Illustration of the example in the proof of Proposition 5.8. Missing edges point from right to left.

Proposition 5.9. MoV_{CO} *satisfies degree-consistency.*

Proof. Let T be a tournament, $x \in CO(T)$, $y \in V(T) \setminus CO(T)$, and $\delta = \text{outdeg}(x) - \text{outdeg}(y)$. We claim that $-\delta \leq \mathsf{MoV}_{CO}(y, T) \leq -(\delta - 1)$. The left inequality follows from the fact that if we reverse δ incoming edges into y, then y becomes a Copeland winner. For the right inequality, note that each edge reversal decreases the difference $\text{outdeg}(x) - \text{outdeg}(y)$ by at most 1; the only exception is the edge (x, y), in which case the difference decreases by 2. Since the difference starts at δ and must be nonpositive in order for y to become a Copeland winner, at least $\delta - 1$ edges must be reversed.

Now, let v, w be arbitrary alternatives in T such that $\text{outdeg}(v) > \text{outdeg}(w)$. We have $w \notin CO(T)$. If $v \in CO(T)$, then $\mathsf{MoV}_{CO}(v, T) > 0 > \mathsf{MoV}_{CO}(w, T)$. Assume that $v \notin CO(T)$. Considering an alternative $u \in CO(T)$, we have

$$\begin{aligned}\mathsf{MoV}_{CO}(v, T) - \mathsf{MoV}_{CO}(w, T) &\geq -(\text{outdeg}(u) - \text{outdeg}(v)) + (\text{outdeg}(u) - \text{outdeg}(w) - 1) \\ &= \text{outdeg}(v) - \text{outdeg}(w) - 1 \\ &\geq 0,\end{aligned}$$

meaning that MoV_{CO} is degree-consistent. $\square$

5.2.2 Top Cycle

Next, we consider the top cycle. Recall that, for a given tournament T of size n, TC coincides with k-kings for $k = n - 1$. In order to show that MoV_{TC} satisfies strong degree-consistency, we need Lemma 3.2 as well as the following lemma, which establishes a surprisingly succinct relation between the sizes of the minimum cuts with respect to a pair of alternatives.[12] For alternatives x, y, denote by $\text{min-cut}_k(x, y)$ the size of a smallest k-bounded x-y-cut, and define $\text{min-cut}(x, y) = \text{min-cut}_{n-1}(x, y)$.

Lemma 5.10. *Let T be a tournament and $x, y \in V(T)$. Then,*

$$\text{min-cut}(x, y) - \text{min-cut}(y, x) = \text{outdeg}(x) - \text{outdeg}(y).$$

Proof. For ease of presentation, we divide the alternatives in $V(T) \setminus \{x, y\}$ into four sets:

- D_x consists of the alternatives dominated by x but not y;
- D_y consists of the alternatives dominated by y but not x;
- D_{xy} consists of the alternatives dominated by both x and y;
- D_0 consists of the alternatives dominated by neither x nor y.

See Figure 11 for an illustration.

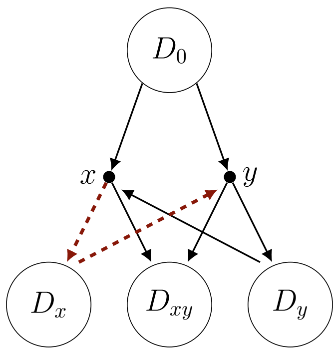

Figure 11: Illustration of the construction used in the proof of Lemma 5.10.

Call a path from x to y an *x-y-path*. From the max-flow min-cut theorem [Ford and Fulkerson, 1956], the size of a minimum cut from x to y equals the maximum number of edge-disjoint x-y-paths (and analogously for a minimum cut from y to x). Making use of this fact, we will argue about maximum sets of edge-disjoint paths instead of minimum cuts. Let $\mathcal{P}_x$ be the set of all paths of length one or two from x to y. Similarly, let $\mathcal{Q}_y$ be the set of all paths of length one or two from y to x.

Claim. There exists a maximum set of edge-disjoint x-y paths, $\mathcal{P}$, such that $\mathcal{P}_x \subseteq \mathcal{P}$, and a maximum set of edge-disjoint y-x paths, $\mathcal{Q}$, such that $\mathcal{Q}_y \subseteq \mathcal{Q}$.

Proof of Claim. By symmetry, it suffices to prove the former statement. Let $\mathcal{P}$ be a maximum set of edge-disjoint x-y paths. We show how we can alter $\mathcal{P}$ in an iterative manner so that $\mathcal{P}_x \subseteq \mathcal{P}$ holds at the end while $\mathcal{P}$ remains a maximum set of edge-disjoint x-y paths.

If $(x, y) \in E(T)$, then also $\{x \to y\} \in \mathcal{P}$, since otherwise $\mathcal{P}$ cannot be maximum. Next, consider some $z \in D_x$, and let $F = \{(x, z), (z, y)\}$. If there exists exactly one path in $\mathcal{P}$ containing an edge from F, we replace this path by the path $x \to z \to y$. Else, if there exist two paths P_1 and P_2 containing an

[12]The lemma also follows from a more general statement by Bubboloni and Gori [2018, Prop. 16]. We thank Daniela Bubboloni and Michele Gori for pointing this out to us.

edge from F, then we can assume without loss of generality that P_1 starts with the edge (x, z) and P_2 ends with the edge (z, y). In this case, we replace P_1 by $x \to z \to y$, and construct P_2 by joining the remaining parts of the two paths to go from x to y through z, possibly omitting any cycles that arise. The newly created paths are edge-disjoint with respect to all other paths in $\mathcal{P}$, and the number of paths in $\mathcal{P}$ remains unchanged. At the end of this process, we have $\mathcal{P}_x \subseteq \mathcal{P}$. $\square$

Using the Claim, we let $\mathcal{P}$ (resp., $\mathcal{Q}$) be a maximum set of edge-disjoint x-y-paths (resp., y-x-paths) such that $\mathcal{P}_x \subseteq \mathcal{P}$ (resp., $\mathcal{Q}_y \subseteq \mathcal{Q}$) holds. We next show that $|\mathcal{P} \setminus \mathcal{P}_x| = |\mathcal{Q} \setminus \mathcal{Q}_y|$. Suppose that this is not the case, and assume without loss of generality that $|\mathcal{P} \setminus \mathcal{P}_x| > |\mathcal{Q} \setminus \mathcal{Q}_y|$. From $\mathcal{P} \setminus \mathcal{P}_x$, we will construct a set of edge-disjoint y-x paths, $\mathcal{Q}'$, which is also edge-disjoint to all paths in $\mathcal{Q}_y$ and is of size $|\mathcal{Q}'| = |\mathcal{P} \setminus \mathcal{P}_x|$, so that $\mathcal{Q}_y \cup \mathcal{Q}'$ contradicts the maximality of $\mathcal{Q}$.

To this end, let $P \in \mathcal{P} \setminus \mathcal{P}_x$. Note that P is of the form $x \to v_1 \to \cdots \to v_\ell \to y$ for some $2 \leq \ell \leq n-2$. Also, $v_1 \in D_{xy}$, since otherwise P would intersect with a path in $\mathcal{P}_x$. For the same reason, $v_\ell \in D_0$. Hence, $(y, v_1), (v_\ell, x) \in E(T)$ and therefore $y \to v_1 \to \cdots \to v_\ell \to x$, where the part between v_1 and v_ℓ is the same as in P, is a y-x-path in T. We create $\mathcal{Q}'$ by using this mirroring argument for all paths in $\mathcal{P} \setminus \mathcal{P}_x$. By construction, the paths in $\mathcal{Q}'$ are edge-disjoint with respect to the paths in $\mathcal{Q}_y$, so $\mathcal{Q}'$ has the desired property. Hence, $|\mathcal{P} \setminus \mathcal{P}_x| = |\mathcal{Q} \setminus \mathcal{Q}_y|$.

Finally, we have

$$\begin{aligned}
\text{min-cut}(x, y) - \text{min-cut}(y, x) &= |\mathcal{P}| - |\mathcal{Q}| \\
&= |\mathcal{P}_x| + |\mathcal{P} \setminus \mathcal{P}_x| - |\mathcal{Q} \setminus \mathcal{Q}_y| - |\mathcal{Q}_y| \\
&= |\mathcal{P}_x| - |\mathcal{Q}_y| \\
&= \text{outdeg}(x) - \text{outdeg}(y),
\end{aligned}$$

as desired. $\square$

Theorem 5.11. MoV_{TC} *satisfies strong degree-consistency.*

Proof. Fix a tournament T and let $x, y \in V(T)$ with $\text{outdeg}(x) \geq \text{outdeg}(y)$. First, we show that $x, y \in TC(T)$ constitutes the only non-trivial case. Since all alternatives in $TC(T)$ dominate all alternatives outside, it cannot be that $x \notin TC(T)$ and $y \in TC(T)$. If $x, y \notin TC(T)$, Theorem 4.2 implies that $\mathsf{MoV}_{TC}(x, T) = -1 = \mathsf{MoV}_{TC}(y, T)$. If $x \in TC(T)$ and $y \notin TC(T)$, then $\mathsf{MoV}_{TC}(x) > 0 > \mathsf{MoV}_{TC}(y)$.

Assume now that $x, y \in TC(T)$. Let R be a minimum DRS for x. By Lemma 3.2 with $k = n-1$, we know that R is a minimum x-t-cut for some $t \in V(T)$. We consider two cases. First, assume that R is also a y-t-cut. Then, a minimum y-t-cut $R' \subseteq E(T)$ satisfies $|R'| \leq |R|$, proving that $\mathsf{MoV}_{TC}(x, T) = |R| \geq |R'| \geq \mathsf{MoV}_{TC}(y, T)$. For the second case, assume that R is not a y-t-cut. Then, R needs to be an x-y-cut (since otherwise x can reach t via y), and therefore it must be a minimum x-y-cut. By Lemma 5.10, since $\text{outdeg}(x) \geq \text{outdeg}(y)$, for a minimum y-x-cut R' it holds that $|R| \geq |R'|$. Hence $\mathsf{MoV}_{TC}(x, T) = |R| \geq |R'| \geq \mathsf{MoV}_{TC}(y, T)$. $\square$

5.2.3 Uncovered Set, k-Kings, and Banks Set

We now consider UC, BA, and k-kings, and show that these tournament solutions do not satisfy any of the degree-consistency properties.

Proposition 5.12. MoV_{UC} *and* MoV_{BA} *do not satisfy equal-degree-consistency.*

Proof. We give a counterexample for both MoV_{UC} and MoV_{BA} at once. The example tournament T contains seven alternatives, a, b, c, d, e, f, g, which all have the same outdegree. See Figure 12 for an illustration.[13]

[13]This tournament has been previously considered by Brandt et al. [2018, Fig. 7].

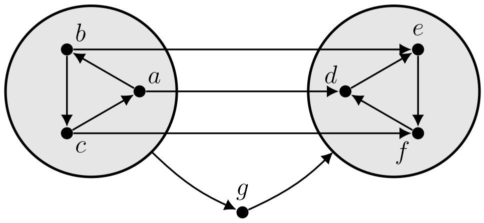

Figure 12: Illustration of the example in the proof of Proposition 5.12. Missing edges point from right to left.

We start by showing that this is a counterexample for MoV_{UC}. Note that all alternatives are in the uncovered set of this tournament. We claim that $\mathsf{MoV}_{UC}(g,T) = 2$ while $\mathsf{MoV}_{UC}(d,T) = 1$. The former claim follows from Lemma 3.2 and the observation that g has exactly two edge-disjoint paths of length at most two to every other alternative. For the latter, note that $d \to b$ is the only path of length at most two from d to b.

Next, we show that the counterexample holds for MoV_{BA} as well. To this end, we show that $g, d \in BA(T)$ and $\mathsf{MoV}_{BA}(g,T) > 1$. Since $BA(T) \subseteq UC(T)$ and therefore $\mathsf{MoV}_{BA}(d,T) \leq \mathsf{MoV}_{UC}(d,T) = 1$, this suffices to proof the claim. Since the transitive subtournament consisting of g, d, e cannot be extended, $g \in BA(T)$. Likewise, $d \in BA(T)$ because the subtournament consisting of d, b, e cannot be extended.

It remains to argue that $\mathsf{MoV}_{BA}(g,T) > 1$. Assume for contradiction that $\mathsf{MoV}_{BA}(g,T) = 1$, i.e., there exists an edge whose reversal takes g out of the Banks set. Suppose first that g still dominates all of d, e, f after the reversal. Then, since each of a, b, c is dominated by two of d, e, f in T, at least one of the three transitive subtournaments with alternative set $\{g, d, e\}, \{g, e, f\}, \{g, f, d\}$ cannot be extended. The remaining case is that g no longer dominates all of d, e, f, meaning that an edge (g, x) is reversed for some $x \in \{d, e, f\}$. Assume without loss of generality that $x = d$. In this case, the transitive subtournament formed by g, e, f still cannot be extended, so $\mathsf{MoV}_{BA}(g,T) > 1$. This concludes the proof. $\square$

Proposition 5.13. MoV_{UC} *and* MoV_{BA} *do not satisfy degree-consistency.*

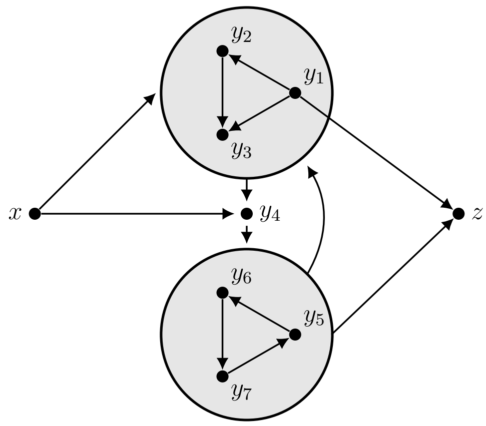

Figure 13: Illustration of the example in the proof of Proposition 5.13. Missing edges point from right to left.

Proof. We give a counterexample for both MoV_{UC} and MoV_{BA} at once. The example tournament T consists of nine alternatives, x, z, and y_i for $i = 1, \dots, 7$. See Figure 13 for an illustration. Alternative x dominates exactly y_1, y_2, y_3 and y_4, and alternative z dominates x, y_2, y_3 and y_4. In general y_i dominates y_j whenever $i < j$, with the exceptions that y_7 dominates y_5, and $\{y_5, y_6, y_7\}$ dominate $\{y_1, y_2, y_3\}$.

We start by showing that this is a counterexample for MoV_{UC}. First, observe that both x and y_4 belong to $UC(T)$. Moreover, x has outdegree 4, and $\mathsf{MoV}_{UC}(x,T) = 1$ by Lemma 3.2 since there is only one path of length at most two from x to z. On the other hand, y_4 has outdegree 3, but its MoV_{UC} is 2. To see this, note that y_4 can reach each alternative in $\{y_5, y_6, y_7\}$ both directly and through another alternative in this latter set. Moreover, since y_5, y_6, y_7 all dominate the remaining alternatives, y_4 has three disjoint paths of length two to each of these alternatives.

Next, we show that the counterexample holds for MoV_{BA} as well. To this end we show that $x, y_4 \in BA(T)$ and $\mathsf{MoV}_{BA}(y_4, T) > 1$. Since $BA(T) \subseteq UC(T)$ and therefore $\mathsf{MoV}_{BA}(x, T) \leq \mathsf{MoV}_{UC}(x, T) = 1$, this suffices to establish the claim. In order to show that $y_4 \in BA(T)$, we define T_{56} to be the subtournament of T induced by the set $\{y_4, y_5, y_6\}$. Analogously, we define T_{67} and T_{75}. It is easy to see that T_{56}, T_{67} and T_{75} are all transitive and y_4 is their maximum element. Moreover, none of them can be extended by any other alternative, meaning that $y_4 \in BA(T)$. In order to show that $x \in BA(T)$, consider the subtournament induced by $\{x, y_1, y_2, y_3, y_4\}$, and observe that it cannot be extended by any other alternative.

It remains to argue that $\mathsf{MoV}_{BA}(y_4, T) > 1$. Assume for contradiction that $\mathsf{MoV}_{BA}(y_4, T) = 1$ and let $\{(a, b)\}$ be a destructive reversal set for y_4, i.e., $y_4 \notin BA(T')$, where T' is obtained from T by reversing the edge (a, b). We perform a case distinction on the identity of a and b. First, consider the cases where $a, b \in \{x, y_1, y_2, y_3, y_4, z\}$ or $a, b \in \{y_5, y_6, y_7\}$. Then, T'_{56}, T'_{67} and T'_{75} (defined analogously as for T) are transitive subtournaments with maximal element y_4 which cannot be extended. Second, let one of a and b be from $\{x, y_1, y_2, y_3, y_4, z\}$ while the other one is from $\{y_5, y_6, y_7\}$, and without loss of generality let $\{a, b\} \cap \{y_5, y_6, y_7\} = \{y_5\}$. Then, the subtournament T'_{67} is still transitive, has y_4 as a maximal element, and cannot be extended. It follows that $y_4 \in BA(T')$, a contradiction to the assumption that $\{(a, b)\}$ is a destructive reversal set. This concludes the proof. $\square$

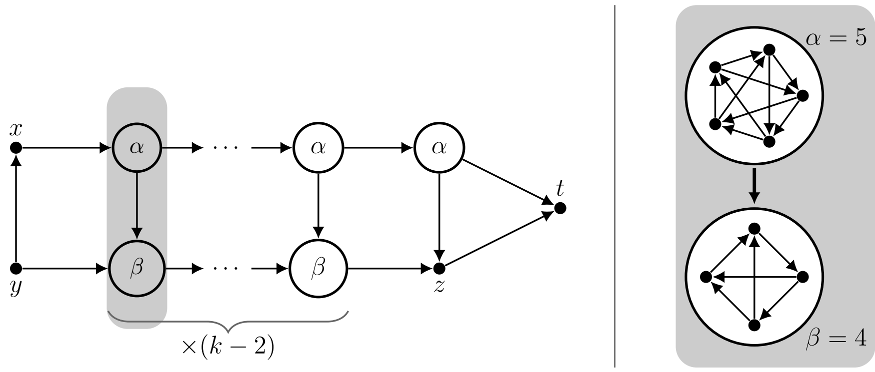

Figure 14: Illustration of the example in the proof of Proposition 5.14. Missing edges point from right to left. The left image gives an overview of the example, while the right image shows a close-up of two "supernodes" of size $\alpha = 5$ and $\beta = 4$, respectively.

Proposition 5.14. $\mathsf{MoV}_{k\text{-kings}}$ *(for constant $k \geq 3$) satisfies neither degree-consistency nor equal-degree-consistency.*

Proof. Let $k \geq 3$ be a constant. We describe a family of examples which, after specifying two parameters, allows us to disprove the degree-consistency as well as the equal-degree-consistency of $\mathsf{MoV}_{k\text{-kings}}$. The high-level idea of the instance is as follows: There exist two alternatives, x and y, both of which are currently k-kings. Moreover, $\mathrm{outdeg}(x) = \alpha$ and $\mathrm{outdeg}(y) = \beta + 1$, where α is any odd positive integer while β can be any positive integer. The example is constructed in such a way that the $\mathsf{MoV}_{k\text{-kings}}$ of x is at least $(\alpha + 1)/2$ while that of y is 1. Setting $\alpha \geq 3$ and $\beta = \alpha - 1$ yields a violation of equal-degree-consistency, and setting $\alpha \geq 3$ and $\beta \geq \alpha$ yields a violation of degree-consistency.

We now describe the construction in more detail; see Figure 14 for an illustration. The tournament T consists of four singleton alternatives, x, y, z and t, and $2k - 3$ *supernodes*. These supernodes are tournaments themselves, where all alternatives in the supernode have the same relation to each alternative outside of the supernode. In Figure 14, we depict supernodes by large circles. In our construction there exist two different types of supernodes: those with parameter α and those with parameter β. Each supernode with parameter α contains α singleton alternatives and have a specific structure. More precisely, the alternatives are arranged on a cycle and each alternative dominates exactly the $(\alpha - 1)/2$ alternatives following it on the cycle. (This structure is called a "cyclone" in Appendix B.2.) We make fewer specifications for supernodes of parameter β and simply require that each of them corresponds to a tournament of size β, but their inner structure can be chosen arbitrarily. For the relationships between alternatives and supernodes, we refer to the left image of Figure 14.

Claim. Let α be an odd positive integer, β be a positive integer, and T be a tournament with parameters α and β as described in Figure 14. Then, $\text{outdeg}(x) = \alpha$, $\text{outdeg}(y) = \beta + 1$, $\mathsf{MoV}_{k\text{-kings}}(x, T) \geq (\alpha + 1)/2$, and $\mathsf{MoV}_{k\text{-kings}}(y, T) = 1$.

Proof of Claim. The outdegrees of x and y follow by construction, and it can be verified that both x and y are k-kings. We start by showing that the $\mathsf{MoV}_{k\text{-kings}}$ of y in the constructed example is 1. By Lemma 3.2, it suffices to show that there exists a k-length bounded y-t-cut of size 1. The edge (z, t) forms such a cut: after deleting it, all paths from y to t have length at least $k + 1$.

Next, we show that the $\mathsf{MoV}_{k\text{-kings}}$ of x is at least $(\alpha + 1)/2$. We do so by arguing that for any $w \in V(T)$, the size of a minimum k-length bounded x-w-cut is at least $(\alpha + 1)/2$. To this end, we give a lower bound on the number of edge-disjoint paths of length at most k from x to w; clearly, any k-length bounded x-w-cut must have size at least this latter number. First, let w be any alternative that is not included in the supernode dominated by x. In this case, there exist at least α disjoint paths from x to w. This is because there exists a path from x to w, containing at least three alternatives, which uses only alternatives from supernodes of size α in its interior (in other words, all alternatives besides x and w belong to such supernodes) and does not use more than one alternative from the same supernode. By construction, such a path gives rise to α edge-disjoint paths from x to w. Second, let w be an alternative in the supernode dominated by x. Then, due to the structure of the supernode, there exist $(\alpha - 1)/2$ disjoint two-step x-w-paths and one direct x-w-path, i.e., the edge (x, w). Hence, the MoV of x is at least $(\alpha + 1)/2$. $\square$

As discussed earlier, this Claim concludes the proof of Proposition 5.14. $\square$

Corollary 5.15. $\mathsf{MoV}_{CO}, \mathsf{MoV}_{UC}, \mathsf{MoV}_{k\text{-kings}}$ *(for constant $k \geq 3$), and* MoV_{BA} *do not fulfill strong degree-consistency.*

5.3 A Probabilistic Result

In this subsection, we establish a simple formula for the MoV of TC and k-kings for $k \geq 4$ that works "with high probability", i.e., the probability that the formula holds converges to 1 as n grows. We assume that the tournament is generated using the *uniform random model*, where each edge is oriented in either direction with equal probability independently of other edges; this model has been studied, among others, by Fey [2008] and Scott and Fey [2012].

Theorem 5.16. *Let $S \in \{TC, k\text{-kings}\}$, where $4 \leq k \leq n - 1$. Assume that a tournament T is generated according to the uniform random model. Then, with high probability, the following holds for all $x \in V(T)$ simultaneously:*

$$\mathsf{MoV}_S(x, T) = \min \left(\text{outdeg}(x), \min_{y \in V(T): y \neq x} \text{indeg}(y) \right).$$

Theorem 5.16 suggests that when tournaments are generated according to the uniform random model, MoV_{TC} and $\mathsf{MoV}_{k\text{-kings}}$ for $k \geq 4$ can likely be computed by a simple formula based on the degrees of the alternatives. In particular, even though the problem is computationally hard for $\mathsf{MoV}_{k\text{-kings}}$ for any constant $k \geq 4$ (Theorem 3.4), there exists an efficient heuristic that correctly computes the MoV value in most cases. In Appendix B.2, we give an example showing that the heuristic is not always correct. More precisely, for any positive integer ℓ, we construct a tournament such that $\{\mathsf{MoV}_{TC}(x, T) \mid x \in V(T)\}$ contains the values $1, 2, \ldots, \ell$ whereas the formula in Theorem 5.16 predicts that all alternatives have the same (arbitrarily large) MoV_{TC} value.

At a high level, to prove this theorem, we first observe that by a result of Fey [2008], it is likely that $S(T) = V(T)$, i.e., all alternatives are chosen by S. In order to remove alternative x from the winner set, one option is to make it a Condorcet loser—this requires $\mathrm{outdeg}(x)$ reversals—while another option is to make another alternative y a Condorcet winner—this requires $\mathrm{indeg}(y)$ reversals. Hence, the left-hand side is at most the right-hand side. To establish that both sides are equal with high probability, we need to show that the aforementioned options are the best ones for making x a non-winner—by Lemma 3.2, this requires making some y unreachable from x in four steps. The intuition behind this claim is that the tournament resulting from the uniform random model is highly connected, with many paths of length at most four from x to y. As a result, if we want to make y unreachable from x, it is unlikely to be beneficial to destroy intermediate edges instead of edges adjacent to x or y.

To prove the theorem, we first state the Chernoff bound, a standard tool for bounding the probability that the value of a random variable is far from its expectation.

Lemma 5.17 (Chernoff bound). *Let $X_1, \ldots, X_k$ be independent random variables taking values in $[0, 1]$, and let $S := X_1 + \cdots + X_k$. Then, for any $\delta \geq 0$,*

$$\Pr[S \geq (1 + \delta)\mathbb{E}[S]] \leq \exp\left(\frac{-\delta^2\mathbb{E}[S]}{3}\right)$$

and

$$\Pr[S \leq (1 - \delta)\mathbb{E}[S]] \leq \exp\left(\frac{-\delta^2\mathbb{E}[S]}{2}\right).$$

Proof of Theorem 5.16. Let $r := n - 1$, and consider the following three events:

(i) $S(T) = V(T)$;

(ii) For every $x \in V(T)$, it holds that $\mathrm{outdeg}(x), \mathrm{indeg}(x) \in [0.49r, 0.51r]$;

(iii) For every pair of disjoint sets $A, B \subseteq V(T)$ such that $|A|, |B| \geq 0.1r$, the number of edges directed from an alternative in A to an alternative in B is at least $0.004r^2$.

We claim that with high probability, all three events occur simultaneously. By union bound, it suffices to prove this claim for each event separately. The claim for event (i) follows from Theorem 1 of Fey [2008], which shows that the Banks set includes all alternatives with high probability in a random tournament, along with the fact that in any tournament, the Banks set is contained in the uncovered set, which is in turn contained in our tournament solution S.

Fix $x \in V(T)$, and let $X_1, \ldots, X_{n-1}$ be indicator random variables that indicate whether x dominates each of the remaining $n - 1$ alternatives or not; X_i takes the value 1 if so, and 0 otherwise. Let $X := \sum_{i=1}^{n-1} X_i$. We have $\mathbb{E}[X_i] = 0.5$ for each i, and so $\mathbb{E}[X] = 0.5r$. By Lemma 5.17, it follows that

$$\Pr[X \geq 0.51r] \leq \exp\left(-\frac{0.02^2 \cdot 0.5r}{3}\right) \leq \exp(-10^{-5}r).$$

Similarly, by applying the other inequality in Lemma 5.17, we have $\Pr[X \leq 0.49r] \leq \exp(-10^{-5}r)$. Taking a union bound over these two events and over all $x \in V(T)$, the probability that $\mathrm{outdeg}(x) \notin$

$[0.49r, 0.51r]$ for some x is at most $2n \cdot \exp(-10^{-5}r) \leq 4r \cdot \exp(-10^{-5}r)$, which converges to 0 as $r \to \infty$ (equivalently, as $n \to \infty$). Since $\mathrm{outdeg}(x) + \mathrm{indeg}(x) = r$ for each x, having $\mathrm{outdeg}(x) \in [0.49r, 0.51r]$ implies $\mathrm{indeg}(x) \in [0.49r, 0.51r]$ as well. This means that event (ii) occurs with high probability.

Next, fix a pair of disjoint sets $A, B \subseteq V(T)$ such that $|A|, |B| \geq 0.1r$. Let t be the number of edges between A and B, and let $Y_1, \ldots, Y_t$ be indicator random variables that indicate whether each edge is oriented from A to B; Y_i takes the value 1 if so, and 0 otherwise. Let $Y := \sum_{i=1}^{t} Y_i$. We have $\mathbb{E}[Y_i] = 0.5$ for each i, and so $\mathbb{E}[Y] = 0.5t$. Writing $t = cr^2$ for some $c \geq 0.01$, it follows by Lemma 5.17 that

$$\begin{aligned}\Pr[Y \leq 0.004r^2] &= \Pr\left[Y \leq \frac{0.004}{0.5c} \cdot \mathbb{E}[Y]\right] \\ &\leq \Pr[Y \leq 0.8 \cdot \mathbb{E}[Y]] \\ &\leq \exp(-0.02 \cdot \mathbb{E}[Y]) \leq \exp(-10^{-4}r^2).\end{aligned}$$

Since there are no more than 2^n choices for each of A and B, by union bound, the probability that event (iii) fails for some pair A, B is at most $2^{2n} \cdot \exp(-10^{-4}r^2) \leq \exp(4r - 10^{-4}r^2)$, which again vanishes for large r. We have therefore established that events (i), (ii), and (iii) occur simultaneously with high probability.

Assume from now on that all three events occur, and let $r \geq 130$. We will show that under these conditions, it always holds that

$$\mathsf{MoV}_S(x, T) = \min\left(\mathrm{outdeg}(x), \min_{y \in V(T): y \neq x} \mathrm{indeg}(y)\right).$$

This suffices to finish the proof of the theorem.

First, since event (i) occurs, $\mathsf{MoV}_S(x, T)$ is positive for every $x \in V(T)$. We claim that for any distinct $x, y \in V(T)$, it holds that

$$\text{min-cut}_k(x, y) = \min\left(\mathrm{outdeg}(x), \mathrm{indeg}(y)\right). \tag{6}$$

If (6) holds, we would have that the size of a minimum k-length bounded x-cut is

$$\begin{aligned}\min_{y \neq x}(\text{min-cut}_k(x, y)) &= \min_{y \neq x}(\min\left(\mathrm{outdeg}(x), \mathrm{indeg}(y)\right)) \\ &= \min\left(\mathrm{outdeg}(x), \min_{y \neq x} \mathrm{indeg}(y)\right).\end{aligned}$$

By Lemma 3.2, this size is equal to the size of a minimum DRS for x with respect to S, i.e., $\mathsf{MoV}_S(x, T)$, where for TC we take $k = n - 1$. To finish the proof, it therefore remains to establish (6).

Fix a pair $x, y \in V$. Observe that the following two sets are k-length bounded x-y-cuts:

- The set of all outgoing edges from x (since x cannot reach any other alternative upon the removal of these edges);
- The set of all incoming edges into y (since y cannot be reached by any other alternative upon the removal of these edges).

The former set has size $\mathrm{outdeg}(x)$ and the latter set has size $\mathrm{indeg}(y)$, implying that $\text{min-cut}_k(x, y) \leq \min\left(\mathrm{outdeg}(x), \mathrm{indeg}(y)\right)$.

Assume now for the sake of contradiction that this inequality is strict, i.e., there exists a k-length bounded x-y-cut R of size less than $\min\left(\mathrm{outdeg}(x), \mathrm{indeg}(y)\right)$. Let $E(x, D(x))$ denote the set of edges between x and its dominion $D(x)$, and let $E(y, \overline{D}(y))$ denote the set of edges between y and its set of dominators $\overline{D}(y)$. Since event (ii) occurs, we have $|E(x, D(x))|, \left|E(y, \overline{D}(y))\right| \in [0.49r, 0.51r]$. Moreover, $|R| \leq 0.51r$. Let $V_x \subseteq D(x)$ be the set of alternatives that x can still directly reach after the removal of R. Similarly, let $V_y \subseteq \overline{D}(y)$ be the set of alternatives that can directly reach y after the removal of R. We consider two cases:

Case 1: R contains at most $0.39r$ edges in each of $E(x, D(x))$ and $E(y, \overline{D}(y))$. This means that $|V_x|, |V_y| \geq 0.1r$. If $V_x \cap V_y \neq \emptyset$, then x can reach y via a path of length two even after the removal of R, a contradiction. So V_x and V_y must be disjoint. Since event (iii) occurs, there are at least $0.004r^2$ edges directed from an alternative in V_x to an alternative in V_y. In addition, from $r \geq 130$ we have $0.004r^2 > 0.51r$, so at least one of these edges is not included in R. It follows that after R is removed, there still exists a path of length three from x to y, a contradiction.

Case 2: R contains at least $0.39r$ edges in either $E(x, D(x))$ or $E(y, \overline{D}(y))$. Assume without loss of generality that it contains at least $0.39r$ edges in $E(x, D(x))$; the other case can be handled analogously. Since $|R| \leq 0.51r$, R contains at most $0.12r$ edges in $E(y, \overline{D}(y))$. So $|V_y| \geq 0.49r - 0.12r = 0.37r$. Now, since $|R| < \min(\text{outdeg}(x), \text{indeg}(y)) \leq \text{outdeg}(x)$, we have $|V_x| \geq 1$. Let z be an arbitrary alternative in V_x, and let $E(z, D(z))$ denote the set of edges between z and its dominion $D(z)$. Let $V_z \subseteq D(z)$ be the set of alternatives that z can reach directly after R is removed. Repeating our argument for V_y, we get $|V_z| \geq 0.37r$.

The rest of the argument in Case 2 mirrors that of Case 1, with V_z taking the role of V_x. If $V_z \cap V_y \neq \emptyset$, then x can reach y via a path of length three even after the removal of R, a contradiction. Else, V_z and V_y are disjoint. Since event (iii) occurs, there are at least $0.004r^2$ edges directed from an alternative in V_z to an alternative in V_y, so at least one of these edges is not included in R. It follows that after R is removed, there still exists a path of length four from x to y, a contradiction.

We have reached a contradiction in both cases, and the proof is complete. □

5.4 Monotonicity

The definition of MoV implies that edge reversals have limited effects on the MoV value of alternatives: If a single edge e of a tournament T is reversed, then $\mathsf{MoV}_S(x, T)$ and $\mathsf{MoV}_S(x, T^e)$ differ by at most 1, unless x is a winner in exactly one of the two tournaments T and T^e (in which case $|\mathsf{MoV}_S(x, T) - \mathsf{MoV}_S(x, T^e)| = 2$). We show that MoV values behave monotonically with respect to edge reversals provided the underlying tournament solution is monotonic, as is the case for all tournament solutions considered in this paper (Proposition 5.4).

Proposition 5.18. *Let S be a monotonic tournament solution and consider two tournaments T and T^e, where $e = (y, x) \in E(T)$. Then,* $\mathsf{MoV}_S(x, T^e) \geq \mathsf{MoV}_S(x, T)$.

Proof. Let T be a tournament and $e = (y, x) \in E(T)$, and let $T' = T^e$. We first consider the case where $x \in S(T)$, i.e., $\mathsf{MoV}_S(x, T) > 0$. By monotonicity of S, it follows that $x \in S(T')$. Suppose for contradiction that $\mathsf{MoV}_S(x, T') < \mathsf{MoV}_S(x, T)$, and let R' be a minimum DRS for x with respect to T'. Then, we will find a DRS R for x with respect to T of size $|R| \leq |R'|$, contradicting the assumption $\mathsf{MoV}_S(x, T) > \mathsf{MoV}_S(x, T') = |R'|$. To this end, define $R = R' \setminus \{e\}$. To see that R is a DRS for x with respect to T, we need to show that $x \notin S(T^R)$. Since R' is a DRS for x with respect to T', we have $x \notin S(T'^{R'})$. In the case $e \in R'$, the tournament $T'^{R'}$ is identical to T^R, and the claim follows. In the case $e \notin R'$, we have $T'^{R'} = T'^R = T^{R \cup \{e\}}$. Since $x \notin S(T^{R \cup \{e\}})$ and S is monotonic, we have $x \notin S(T^R)$.

An analogous argument can be applied in the case where $x \in V(T) \setminus S(T)$, i.e., $\mathsf{MoV}_S(x, T) < 0$. If $x \in S(T')$, then $\mathsf{MoV}_S(x, T') > 0 > \mathsf{MoV}_S(x, T)$ holds trivially. Therefore, we assume that $x \in V(T) \setminus S(T')$, so $\mathsf{MoV}_S(x, T') < 0$. Suppose for contradiction that $\mathsf{MoV}_S(x, T') < \mathsf{MoV}_S(x, T)$

and let R be a minimum CRS for x with respect to T. Then, by monotonicity, $R' = R \setminus \{e\}$ is a CRS for x with respect to T', contradicting the assumption that $\mathsf{MoV}_S(x, T') < \mathsf{MoV}_S(x, T) = -|R|$. $\square$

Note that Proposition 5.18 clearly fails when S is non-monotonic. Indeed, if strengthening an alternative x takes it out of the winner set, then the MoV of x goes from positive to negative upon the corresponding edge reversal.

6 Experiments

In order to better understand how MoV values of tournament solutions behave in practice, we conducted computational experiments using randomly generated tournaments. For the sake of diversity of the generated instances, we implemented six well-studied stochastic models to generate tournaments. In particular, to make our study comparable to the experiments presented by Brandt and Seedig [2016], we selected a similar set of stochastic models and parameterizations.

Given a tournament solution S and a tournament T, we are interested in

- the number $|\arg\max_{x \in V(T)} \mathsf{MoV}_S(x, T)|$ of alternatives with maximum MoV_S value, and
- the number $|\{\mathsf{MoV}_S(x, T) : x \in V(T)\}|$ of different MoV values taken by all alternatives in the tournament.

The first value directly measures the discriminative power of the refinement of S that only selects alternatives with a maximal MoV_S value, whereas the second value measures more generally the ability of the MoV notion to distinguish between the alternatives in a tournament.

6.1 Setup

We used six stochastic models to generate preferences: the uniform random model (which was used in Section 5.3), two variants of the *Condorcet noise model* (with and without voters), the *impartial culture* model, the Pólya-Eggenberger *urn model*, and the *Mallows* model.

We first describe two models that directly create tournaments without creating a preference profile of a set of voters beforehand. The simplest way to create a tournament is to start with a complete undirected graph and decide the direction of each edge independently by flipping a fair coin—we call this the *uniform random model*. The *Condorcet noise model* is similar but slightly more biased: Here, we start with an initial order $\succ$ on the alternatives and some fixed parameter $1/2 \leq p \leq 1$. Then, for two different alternatives a and b where $a \succ b$, the edge (a, b) is included in the tournament with probability p; otherwise, the edge (b, a) is included.

For the remaining four stochastic models, we first create a preference profile of a set of voters, i.e., each voter has a complete and antisymmetric (but not necessarily transitive) preference relation over the set of alternatives. Like Brandt and Seedig [2016], we set the number of voters to 51. Then, we consider the majority relation, which induces a tournament when there are an odd number of voters. One way to generate a preference profile is similar to the previously discussed Condorcet noise model, i.e., the *Condorcet noise model with voters*. Again, we start with a random order $\succ$ on the alternatives and some fixed parameter $1/2 \leq p \leq 1$. Now, the preference relation for each voter is created just as we created the tournament in the Condorcet noise model, before we take the majority among these preferences.

The other three stochastic models all assign a ranking to each voter, i.e., the individual preference relations are now required to be transitive. In the *impartial culture* model, for each voter, the probability of obtaining a ranking is uniformly distributed over all possible rankings and thereby independent of the selection for other voters. A similar but more correlated way to select rankings is the Pólya-Eggenberger *urn model*, suggested by Berg [1985]. For this model, imagine an urn which initially contains each possible ranking exactly once. Then, after each voter has drawn a ranking from the urn, the ranking is placed back together with α copies of it. Naturally, the parameter α controls the degree of similarity among the voters. Lastly, we also applied the *Mallows* model [Mallows, 1957]. Assuming a ground truth ranking, the probability that a voter is assigned a particular ranking in this model grows when the Kendall tau distance to the ground truth ranking becomes smaller. The dispersion parameter $\phi \in (0, 1]$ controls the concentration of the probability mass on rankings that are close to the ground truth ranking. More precisely, $\phi = 1$ corresponds to the uniform distribution over all possible rankings, while $\phi \to 0$ concentrates more probability on the ground truth ranking and rankings close to it.

For each stochastic model and each number of alternatives $n \in \{5, 10, 15, 20, 25, 30\}$, we sampled 100 tournaments. Using the methods described in Section 3, we implemented algorithms to calculate the MoV values for CO, UC, 3-kings, and TC. Due to their computational intractability, we did not implement procedures to calculate the MoV values for BA and k-kings for $k \geq 4$.

The experiments were carried out on a system with 1.4 GHz Quad-Core Intel Core i5 CPU, 8GB RAM, and macOS 10.15.2 operating system. The software was implemented in Python 3.7.7 and the libraries networkx 2.4, matplotlib 3.2.1, numpy 1.18.2, and pandas 1.0.3 were used. For implementing the Mallows and urn models, we utilized implementations contributed by Mattei and Walsh [2013].

The code for our implementation can be found at http://github.com/uschmidtk/MoV.

6.2 Results and Observations

Figure 15 depicts the average size of the set of alternatives with maximum MoV value, and Figure 16 shows the average number of unique MoV values.

The first observation we make is that $\mathsf{MoV}_{\text{3-kings}}$ behaves rather similarly to MoV_{TC}: the average number of alternatives with maximum MoV grows as n increases, and this number is on average slightly less than half of the number of 3-kings and TC winners, respectively. However, this ratio becomes smaller for tournaments where the number of 3-kings or TC winners is already large. For example, when we only consider tournaments where the number of TC winners is greater than 10, only one-third of the TC winners have a maximum MoV_{TC} value on average; the same holds for 3-kings. Nevertheless, a more detailed look at the experimental results show that for both 3-kings and TC, the set of alternatives with maximum MoV consists of only one alternative in around 73% of all instances, while in the remaining instances this set is typically large. This particular behavior for TC and the uniform random model can be partially explained by Theorem 5.16:[14] With high probability, the MoV values for TC winners follow a specific formula based on the degrees, which leads to the set of alternatives with maximum MoV containing either a single alternative or a large number of alternatives in most cases.[15] Our experiments show that this behavior is also present in tournaments generated by other stochastic models as well as for 3-kings; formalizing the behavior theoretically is an interesting future direction.

Our second main observation is that MoV_{UC} behaves quite differently from $\mathsf{MoV}_{\text{3-kings}}$ and MoV_{TC}. Most importantly, the number of UC winners with maximum MoV_{UC} does *not* increase with a growing number of alternatives, but remains more or less constant for each stochastic model. For the uniform random model and the Condorcet noise models, this value is around 2, while it is roughly 1.4 for Mallows, the urn model, and the impartial culture model. As can be seen in Figure 15, the set of alternatives maximizing MoV_{UC} is almost as discriminative as the Copeland set (all of whose alternatives maximize MoV_{CO}). However, we observe in Figure 16 that the number of unique values of the Copeland score is notably higher than that of MoV_{UC}. The latter is particularly low for models which tend to create tournaments with small UC, including Mallows, impartial culture and the urn model. Both of these

[14] Strictly speaking, the proof of Theorem 5.16 takes $n \geq 131$, but we expect the formula in the theorem statement to already hold with rather high probability for much smaller n.

[15] Indeed, if there is a unique Copeland winner, that winner will be the unique alternative with the largest MoV according to the formula. Otherwise, for several alternatives (including the Copeland winners), it can be the case that their MoV is equal to $\mathrm{indeg}(y)$ for a Copeland winner y.

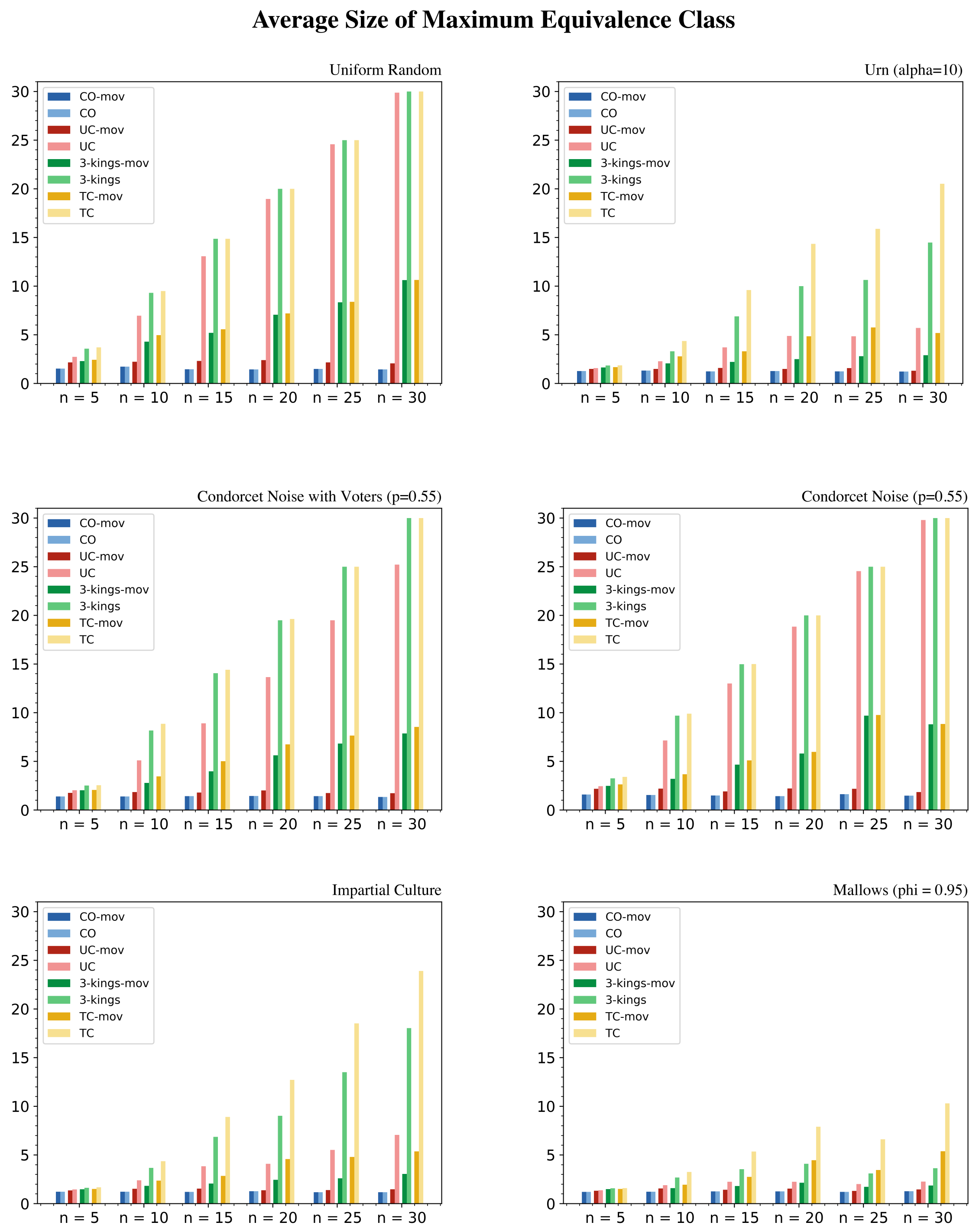

Figure 15: The illustrations show the average number of alternatives with maximum MoV value for different stochastic models, tournament solutions, and sizes. For comparison, the average size of the entire winning set of the corresponding original tournament solution is depicted by a lighter shade.

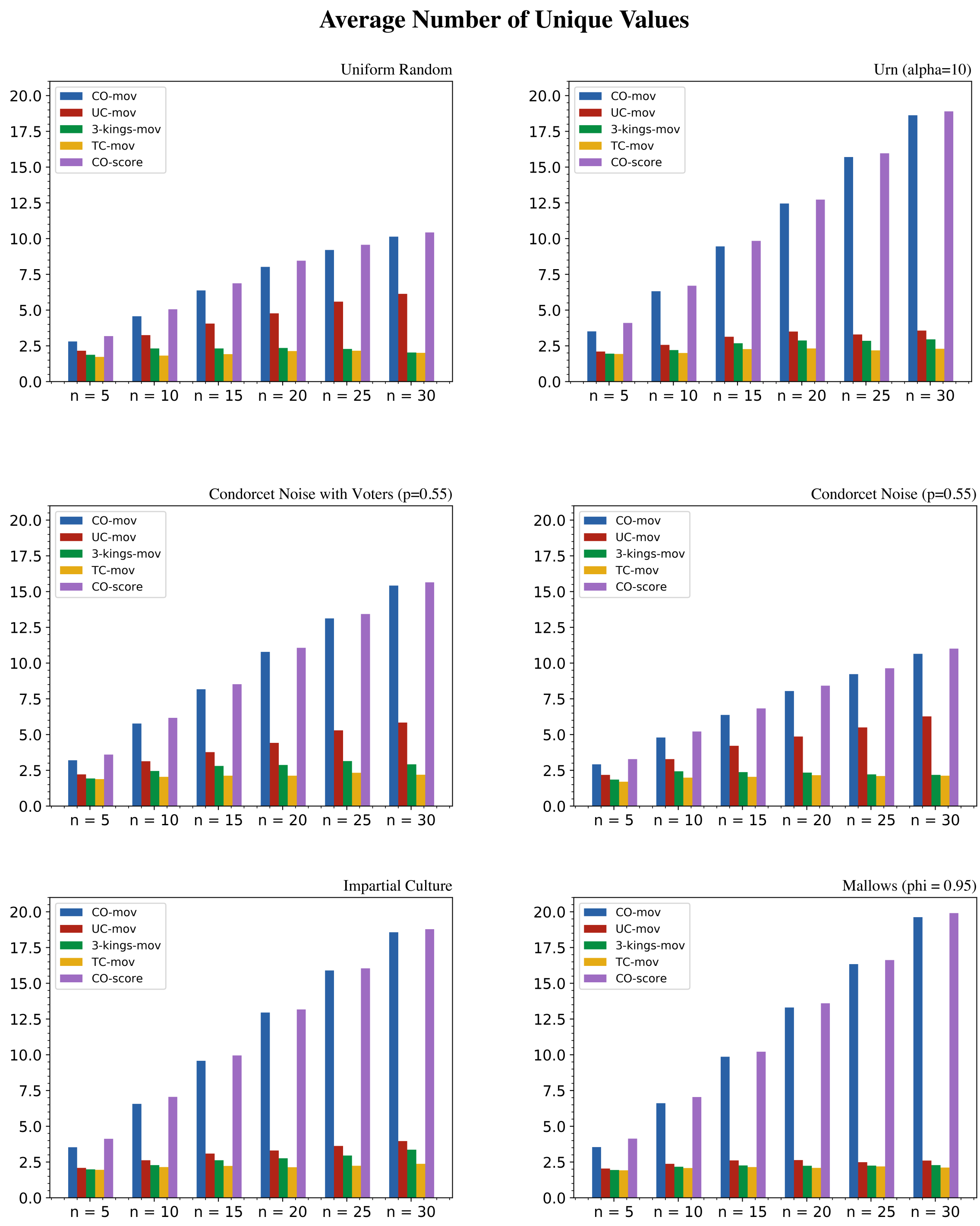

Figure 16: The illustrations show the average number of unique MoV values for different stochastic models, tournament solutions, and sizes. For comparison, the average number of unique Copeland scores is shown in violet.

effects can be explained by the observation that MoV_{UC} is significantly better at distinguishing between UC winners than it is at distinguishing between UC non-winners.[16] As a consequence, tournaments with a small uncovered set generally give rise to a small number of unique MoV_{UC} values.

7 Conclusion and Future Work

In this paper, we have proposed and extensively studied a generic framework for refining tournament solutions based on the notion of margin of victory (MoV). We determined the complexity of computing the MoV, as well as bounds on the MoV, for several common tournament solutions. Moreover, we provided not only structural insights such as consistency of the MoV with respect to the covering relation and Copeland scores, but also experimental evidence regarding the extent to which the MoV refines winner sets in stochastically generated tournaments. Besides the tournament solutions that we have considered, it would be interesting to study the MoV with respect to other tournament solutions such as the bipartisan set, the Slater set, the Markov set, and the minimal covering set. In Appendix C, we provide a preliminary set of results for these four tournament solutions and describe some challenges that we encountered while trying to extend our results.

Viewing the MoV as a robustness measure, one could aim to obtain more comprehensive information about the space of all (not necessarily minimum) reversal sets. For example, one may ask *how many* reversal sets of cost at most c exist for a given alternative. In particular, one could use the number of minimum reversal sets as a tie-breaker for alternatives with equal MoV. Indeed, for some tournaments, especially small ones, the MoV in the unweighted setting may not distinguish between all winners (or non-winners), as witnessed by our experimental results. A specific example is the tournament in Figure 1, where three of the four uncovered set (UC) winners have a MoV of 1. In this example, c has two minimum reversal sets ($\{(c, f)\}, \{(f, d)\}$), d has four ($\{(d, c)\}, \{(d, b)\}, \{(c, f)\}, \{(b, e)\}$), and e has three ($\{(c, f)\}, \{(e, d)\}, \{(e, c)\}$). Investigating the complexity of computing these numbers is an appealing direction for future work; similar counting questions have been considered in the context of elections [Hazon et al., 2012, Baumeister and Hogrebe, 2020, Boehmer et al., 2020] and knockout tournaments [Aziz et al., 2018].

In our experiments, the MoV function corresponding to UC stands out for its discriminative power: not only is the set "max-MoV_{UC}" (containing all alternatives with maximum MoV_{UC} value) consistently small, but the number of distinct MoV_{UC} values is also relatively high in general. It is consequently tempting to suggest max-MoV_{UC} as a new tournament solution. Besides its discriminative power and structural appeal, it can be computed efficiently (Corollary 3.3) and inherits Pareto optimality from the uncovered set, which it refines [Brandt et al., 2016b]. Another intriguing avenue for future research is therefore to analyze max-MoV_{UC}, as well as max-MoV_S for other tournament solutions S, from an axiomatic perspective.

Acknowledgments

This work was partially supported by the Deutsche Forschungsgemeinschaft under grant BR 4744/2-1, by the European Research Council (ERC) under grant number 639945 (ACCORD), and by an NUS Start-up Grant. We would like to thank Felix Brandt and Piotr Faliszewski for helpful discussions, and the anonymous reviewers of AAAI 2020, AAAI 2021, and Artificial Intelligence Journal for valuable feedback.

[16] Theorem 4.4 shows that the smallest MoV_{UC} value in a tournament is bounded below by $-\lceil \log_2 n \rceil$, and that this bound is asymptotically tight. In our experiments, we observed that in most generated tournaments, the smallest MoV_{UC} value is much higher than this lower bound, namely either -1 or -2.

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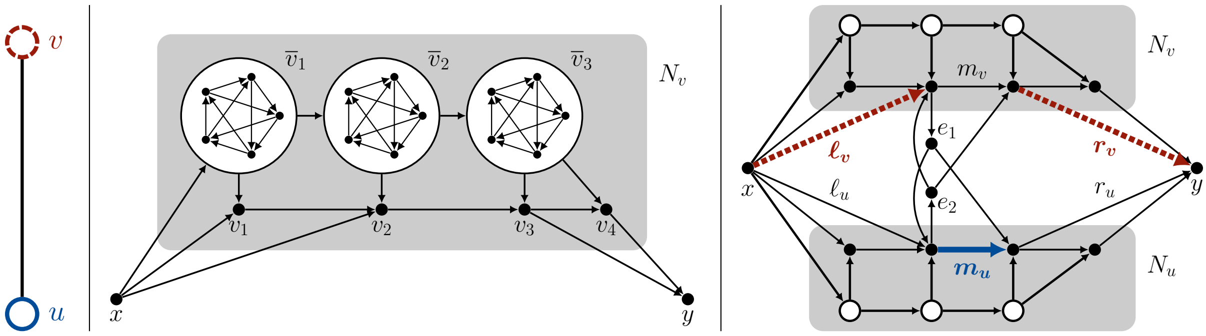

Figure 4: Illustration of the construction used in the proof of Theorem 3.4 for the case $k = 4$. For any undirected graph G (left image), a tournament T is constructed by introducing node gadgets and edge gadgets as follows. A *node gadget* N_v consists of four nodes v_1, v_2, v_3, v_4 and three "supernodes" $\overline{v}_1, \overline{v}_2, \overline{v}_3$, where the latter are tournaments themselves. The center image shows the node gadget for node v. An *edge gadget* for $e = \{u, v\}$ consists of two nodes e_1, e_2 and edges connecting the node gadgets of u and v; see the right image. Nodes x and y are connected to all node gadgets as illustrated. All omitted edges point "backwards" (from right to left) and the direction of vertical edges, if not specified, can be chosen arbitrarily. (repeated from page 11)

A Proof of Theorem 3.4

We reduce from the vertex cover problem. Given an undirected graph $G = (V, E)$, a subset of the nodes $U \subseteq V$ is called a *vertex cover* if for each edge in E, at least one of its endnodes is contained in U. The problem is to determine the minimum cardinality of a vertex cover.

The proof is divided into three parts. We start by showing the reduction for $k = 4$, then extend the construction to arbitrary constants k, and finally argue that we can still carry out the reduction even when we restrict ourselves to cases in which $k \geq n^{1-\epsilon}$ for arbitrarily small constant $\epsilon > 0$.

Part I. Let $k = 4$. From a given instance G of the vertex cover problem, we construct an instance (T, x) of the MoV for 4-kings problem as follows. (An illustration of the construction can be found in the three images of Figure 4.) For ease of presentation we define $n_G := |V(G)|$. For every node $v \in V(G)$ we introduce a *node gadget*, indicated by a grey box in Figure 4 and consisting of four nodes v_1, v_2, v_3 and v_4 as well as three *supernodes*, $\overline{v}_1, \overline{v}_2$ and $\overline{v}_3$. A supernode is itself a tournament consisting of $2n_G + 1$ nodes which are arranged within a circle such that each node has outgoing edges towards the next n_G nodes on the circle and ingoing edges from all other nodes. See the center image in Figure 4 for a close-up of a node gadget with $n_G = 2$. Moreover, we introduce two nodes x and y. Corresponding to the node gadget there exist edges

$$(x, v_1), (x, v_2), (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, y), (v_3, y)$$

as well as *superedges* (i.e., edges that connect at least one supernode)

$$(x, \overline{v}_1), (\overline{v}_1, \overline{v}_2), (\overline{v}_2, \overline{v}_3), (\overline{v}_1, v_1), (\overline{v}_2, v_2), (\overline{v}_3, v_3), (\overline{v}_3, v_4).$$

More precisely, a superedge $(\overline{u}, \overline{v})$ is a set of edges going from all nodes in $\overline{u}$ to all nodes in $\overline{v}$. We extend the node gadget to a tournament by letting all unspecified edges point "backwards" (from right to left) according to Figure 4. In case two nodes are on the same vertical line and the edge between them has not been specified previously, it can be chosen arbitrarily.

For every edge $e = (u, v) \in E$ we introduce an *edge gadget*, consisting of two nodes e_1 and e_2 and the (super)edges $(v_2, e_1), (e_1, u_2), (e_1, u_3), (\overline{v}_2, e_1)$ and analogously $(u_2, e_2), (e_2, v_2), (e_2, v_3), (\overline{u}_2, e_2)$. For the sake of clarity we omit the superedges $(\overline{v}_2, e_1)$ and $(\overline{u}_2, e_2)$ in the illustration of Figure 4. For all nodes $w \in V(G) \setminus \{u, v\}$ we add edges (e_1, w_2) and (e_2, w_2). Finally, all edges unspecified thus far in the entire construction point backwards and the direction of non-specified vertical edges can be chosen arbitrarily.

Lemma 3.2 implies that the cardinality of a minimum 4-bounded x-cut is equal to the cardinality of a minimum DRS for x with respect to 4-kings. Hence, showing the following claim suffices to prove the theorem for $k = 4$.

Claim. For $c \leq n_G$, there exists a vertex cover of size c in G if and only if there exists a 4-bounded x-cut of size $c + n_G$ in T.

Proof of Claim. We start by showing the implication from left to right. Let $U \subseteq V(G)$ be a vertex cover in G. For every node $v \in V(G)$ we name the following three edges inside the node gadget $\ell_v := (x, v_2), m_v := (v_2, v_3)$ and $r_v := (v_3, y)$; the first and last are depicted by red dashed edges in Figure 4. We construct the edge set $C \subseteq E(T)$, which we will show is a 4-bounded x-y-cut, by iterating over all $v \in V(G)$: If $v \in U$, we choose ℓ_v and r_v to be in the set C. On the other hand, if $v \notin U$, we include m_v in the set C. It is easy to see that $|C| = n_G + c$, so it remains to show that C is a 4-bounded x-y-cut.

Since the tournament T contains a very large number of x-y-paths, we argue in the following that, when focusing on $\mathcal{P}_{x,y}(4)$, we can virtually restrict ourselves to the set of "visible" paths in Figure 4, i.e., paths that do not contain omitted edges. See Table 4 and Table 5 which contain distances from x and to y within the visible subgraphs of T, respectively. The columns of the tables correspond to the positions of the nodes within the horizontal alignment in Figure 4.

Pos. 1	Pos. 2	Pos. 3	Pos. 4
$\overline{v}_1 \mid 1$	$\overline{v}_2 \mid 2$	$\overline{v}_3 \mid 3$	
$v_1 \mid 1$	$v_2 \mid 1$	$v_3 \mid 2$	$v_4 \mid 3$
	$e_1 \mid 2$		

Table 4: Distances from x to nodes in node gadget N_v.

Pos. 1	Pos. 2	Pos. 3	Pos. 4
$\overline{v}_1 \mid 4$	$\overline{v}_2 \mid 3$	$\overline{v}_3 \mid 2$	
$v_1 \mid 3$	$v_2 \mid 2$	$v_3 \mid 1$	$v_4 \mid 1$
	$e_1 \mid 2$		

Table 5: Distances from nodes in node gadget N_v to y.

We first claim that the introduction of backward and vertical edges does not change these distances. To see this, let v_i and v_j be (super)nodes at position i and j respectively, where $i \leq j$. For any choice of v_i and v_j it holds that $dist(x, v_j) + 1 \geq dist(x, v_i)$ and $dist(v_i, y) + 1 \geq dist(v_j, y)$, where $dist()$ denotes the distance between two nodes. Hence the backward and vertical edges cannot decrease these distances.

Second, we claim that no backward edge is included in a path in $\mathcal{P}_{x,y}(4)$. Assume for contradiction that there exists such a path with backward edge e. First, assume that the head of e is at position 1. Observe that e cannot be the first edge in the path. However, from Table 5, the minimum distance from position 1 to y is 3, a contradiction. Second, assume that the head of e is at position 2 and observe that e needs to be the third or fourth edge in the path since the path needs to reach position 3 or 4 before using edge e, according to Table 4. However, the minimum distance from position 2 to y is 2, a contradiction. Lastly, assume that the head of edge e is in position 3 and observe that edge e needs to already be the fourth edge in the path since position 4 cannot be reached in less than 3 steps, a contradiction.

Third, we claim that there exist exactly two types of omitted vertical edges that are included in paths in $\mathcal{P}_{x,y}(4)$. A vertical edge (u, v) is included in a path in $\mathcal{P}_{x,y}(4)$ if and only if $dist(x, u) + dist(v, y) \leq 3$. Looking at Table 4 and 5 we see that the only candidates for such an edge are $\{v_2, u_2\}$, $\{v_2, e_1\}$, $\{v_2, e_2\}$ and $\{v_3, u_3\}$. Since the edges (v_2, e_1) and (e_2, v_2) have already been specified, there remain only $\{v_2, u_2\}$ and $\{v_3, u_3\}$. No matter how the directions of these edges are chosen, they create two new paths of length four, e.g., $\{\ell_v, (v_2, u_2), m_u, r_u\}$ and $\{\ell_u, m_u, (u_3, v_3), r_v\}$.

Lastly, we claim that no node contained in a supernode can be used in any path of $\mathcal{P}_{x,y}(4)$. To see this, note that the distances from x and to y sum up to five for all supernodes.

With all this in mind, we show that every path in $\mathcal{P}_{x,y}(4)$ includes some edge in C. First, consider all paths in $\mathcal{P}_{x,y}(4)$ which only use edges within one node gadget, say the one corresponding to $v \in V(G)$. These are exactly the paths $\{\ell_v, m_v, r_v\}$, $\{\ell_v, m_v, (v_3, v_4), (v_4, y)\}$ and $\{(x, v_1), (v_1, v_2), m_v, r_v\}$. All of these paths contain m_v and at least one of the edges ℓ_v and r_v and hence, independently of whether $v \in U$ or not, the paths include an edge in C. Second, consider paths in $\mathcal{P}_{x,y}(4)$ which use edges within two node gadgets u and v which are not neighboring in the graph G. How these paths look exactly depends on the direction of the edge betweeen u_2 and v_2 as well as the edge between u_3 and v_3. Without loss of generality, assume that they are $\{\ell_v, (v_2, u_2), m_u, r_u\}$ and $\{\ell_u, m_u, (u_3, v_3), r_v\}$. These paths have the property that for one of the nodes, say u, they contain either both ℓ_u and m_u, or both m_u and r_u. Hence, independently of whether $u \in U$ or not, the paths include an edge in C. Lastly, consider paths in $\mathcal{P}_{x,y}(4)$ which use two node gadgets corresponding to neighboring nodes in G, say u and v. These paths are $\{\ell_v, (v_2, e_1), (e_1, u_3), r_u\}$ and $\{\ell_u, (u_2, e_2), (e_2, v_3), r_v\}$ as well as (with the same without-loss-of-generality assumption as earlier in this paragraph), $\{\ell_v, (v_2, u_2), m_u, r_u\}$ and $\{\ell_u, m_u, (u_3, v_3), r_v\}$. All of them have the property that they contain either both ℓ_v and r_u, or both ℓ_u and r_v. Since U is a vertex cover, we know that at least one of the pairs ℓ_v, r_v and ℓ_u, r_u is included in C, and therefore the paths contain an edge in C. We summarize that C is a 4-bounded x-y-cut of size $|U| + n_G$.

We turn to prove the direction of the Claim from right to left. Let $C \subseteq V$ be a 4-bounded x-cut of size $n_G + c$ with $c \leq n_G$. For any node $z \in V(T) \setminus \{y\}$, the set C cannot be a 4-bounded x-z-cut because there exist at least $2n_G + 1 > n_G + c$ disjoint 4-bounded x-z-paths due to the introduction of the supernodes. Hence, C is a 4-bounded x-y-cut. In the following, we transform C so that it only contains edges of type ℓ_v, m_v and r_v.

In our first step, we ensure that no edges connecting node gadgets or having an endnode from an edge gadget are included in C. Assume that C contains an edge e connecting gadgets corresponding to the nodes u and v, where we consider e_2 to correspond to node u and e_1 to node v. If e is not included in any path of $\mathcal{P}_{x,y}(4)$, we simply delete e from C. Otherwise, e is contained in exactly one path from $\mathcal{P}_{x,y}(4)$. (To see this, consult the complete characterization of $\mathcal{P}_{x,y}(4)$ in the previous part of the proof.) Moreover, this path contains exactly one of ℓ_u and ℓ_v. We replace e by the edge ℓ_u or ℓ_v, respectively, and obtain another 4-bounded x-y-cut of the same or smaller size.

In our second step, we guarantee that within the node gadget for each node v, either edge m_v or both of the edges ℓ_v and r_v are selected. If one edge from this gadget is selected, it must be the edge m_v, since otherwise there exists at least one path in $\mathcal{P}_{x,y}(4)$ which does not contain an edge in C. If two or more edges are selected, we instead select edges ℓ_v and r_v, since all paths that contain an edge from the node gadget either contain ℓ_v or r_v. Hence, we obtain a new 4-bounded x-y-cut of size at most the size of the previous cut.

After the transformation of C, we derive a vertex cover U of size c in the graph G. For each node $v \in V(G)$ we include v in U if and only if ℓ_v and r_v are included in C. Clearly, $|U| = |C| - n_G$. In case we previously reduced the cardinality of C, we simply add nodes to U until $|U| = c$. Now, assume for contradiction that U is not a vertex cover, i.e., there exists an edge $\{u, v\} \in E(G)$ such that $u, v \notin U$. This means that C contains both m_u and m_v and no other edge from the node gadgets of u and v. Then the path $\{\ell_v, (v_2, e_1), (e_1, u_3), r_u\}$ does not contain an edge in C, a contradiction to C

being a 4-bounded x-y-cut. □

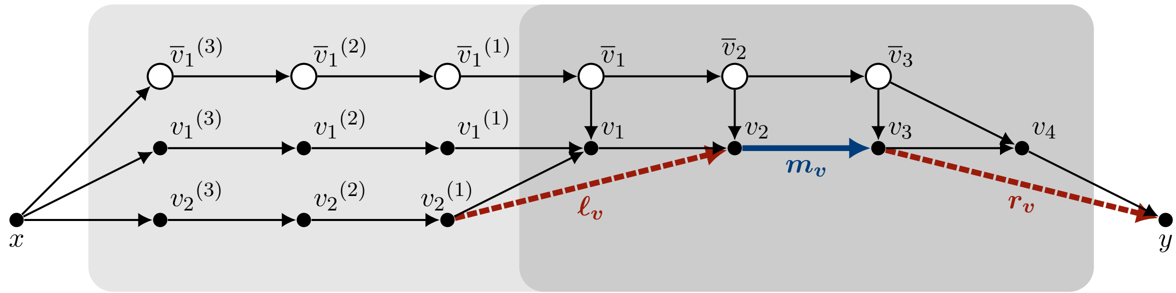

Figure 18: Example illustration of the extended node gadget for $k = 7$ as introduced in the proof of Theorem 3.4.

Part II. We now show how we can adjust the construction for fixed $k > 4$. We change the definition of a node gadget by extending it with a set of preceding (super)nodes

$$\{\overline{v}_1{}^{(i)}, v_1{}^{(i)}, v_2{}^{(i)} \mid \forall\, i \in \{1, \dots, k-4\} \text{ and } \forall\, v \in V(G)\}.$$

(Super)edges go from x to all nodes with superscript $k-4$ and more generally from a node with superscript $i+1$ to the node of the same type with superscript i if they correspond to the same node $v \in V(G)$. Moreover, a superedge points from $\overline{v}_1{}^{(1)}$ towards $\overline{v}_1$, an edge from $v_1^{(1)}$ towards v_1, and edges from $v_2^{(1)}$ towards v_1 and v_2. For the connections among the nodes $v_1, v_2, v_3, v_4, \overline{v}_1, \overline{v}_2, \overline{v}_3, y$ we use the same edges as in the case $k = 4$. Edge gadgets are defined exactly as in the case $k = 4$. All non-specified edges point backwards and vertical edges can be chosen arbitrarily. For every $v \in V(G)$ we define $\ell_v := (v_2^{(1)}, v_2)$, $m_v := (v_2, v_3)$ and $r_v := (v_3, y)$. See Figure 18 for an illustration of the extended node gadget. We show the same claim as previously and omit analogous arguments.

Claim. For $c \leq n_G$, there exists a vertex cover of size c in G if and only if there exists a k-bounded x-cut of size $c + n_G$ in T.

Proof of Claim. We start by showing the implication from left to right. Let $U \subseteq V(G)$ be a vertex cover. Similarly to before, we construct $C \subseteq E(T)$ by choosing for every $v \in V(G)$ the edges ℓ_v and r_v whenever $v \in U$ and the edge m_v when $v \notin U$. While it is easy to see that $|C| = |U| + n_G$, we need to show that C is a k-bounded x-y-cut. Hence, we are interested in the set $\mathcal{P}_{x,y}(k)$ and need to show that C intersects each of its paths at least once. We compare the set $\mathcal{P}_{x,y}(k)$ to the set of 4-bounded x-y-paths which we characterized in the proof for the claim for $k = 4$. To this end, let T_4 be the tournament that we constructed in Part I of the proof. First, consider only the "visible" subgraph of T and note that, in comparison to the graph T_4, the nodes $v_1, v_2, v_3, v_4, e_1, \overline{v}_1, \overline{v}_2$ and $\overline{v}_3$ have a distance from x which is increased by exactly $k - 4$ while the distance to y is the same as previously. Inserting backwards and vertical edges does not change these distances, due to the same arguments as before. Since the rest of the structure of the node gadget is equivalent, the subpaths of the paths in $\mathcal{P}_{x,y}(k)$ that are within the original node gadget (depicted by a darker grey box in Figure 18) correspond to the 4-bounded x-y-paths in T_4. By the same arguments as in Part I, C is a k-bounded x-y-cut.

We turn to prove the implication from right to left. Let C be an x-cut of size $c + n_G$ with $c \leq n_G$. Analogously to the case $k = 4$, we show that we can modify C so that it only contains edges of type ℓ_v, m_v and r_v and is still an x-cut with no greater cost. First, note that for every $z \in V(T) \setminus \{y\}$, there exist at least $2n_G + 1 > n_G + c$ disjoint k-bounded x-z-paths. For example, for the node $v_1^{(3)}$ in Figure 18, this is true because of paths through the supernodes $\overline{v}_1^{(3)}$ and $\overline{v}_1^{(2)}$ and backward edges from $\overline{v}_1^{(2)}$ to $v_1^{(3)}$. Hence, C is in particular an x-y-cut.

In our first step, we ensure that no edges connecting node gadgets or having an endnode from an edge gadget are included in C. We follow a very similar argument as in Part I. First, assume that C contains an edge e connecting gadgets corresponding to the nodes u and v, where we consider e_2 to correspond to node u and e_1 to node v. If e is a backwards edge, it is not contained in any path in $\mathcal{P}_{x,y}(k)$ and we simply delete it from C. If e is a vertical edge, then e is included in at most one path in $\mathcal{P}_{x,y}(k)$. This is due to the fact that when moving from the gadget of u to the gadget of v, the subpath from x to the first node that is reached in node gadget v is necessarily one step longer than the shortest possible path. Hence, the rest of the path is completely determined, as it needs to choose the unique shortest path from x to e as well as the unique shortest path from e to y in order to fulfill the length bound. Moreover, this unique path contains either edge ℓ_u or edge ℓ_v, and we replace edge e by ℓ_u or ℓ_v, respectively.

The second step, in which we guarantee that for every node gadget either the pair ℓ_v, r_v or the edge m_v is chosen, proceeds analogously to the case $k = 4$. We complete the proof by the same argument as before, showing that we can translate C into a vertex cover $U \subseteq E(G)$ of size $|U| = c$. ☐

Part III. It remains to argue that even if we restrict ourselves to the problem with $k \geq n^{1-\epsilon}$ for any fixed $\epsilon > 0$, we can still carry out the previously explained reduction in polynomial time. Let $\epsilon > 0$ be given and the size of the vertex cover instance G be denoted by $n_G := |V(G)|$ and $m_G := |E(G)|$. Moreover, we define n_k to be the number of nodes of the tournament that we construct for a given k. We obtain

$$\begin{aligned} n_k &= (k-1)(2n_G+1)n_G + (2k-4)n_G + 2m_G + 2 \\ &= (2n_G^2 + 3n_G)k + (-2n_G^2 - 5n_G + 2m_G + 2) \\ &\leq \alpha k, \end{aligned}$$

where $\alpha := (2n_G^2 + 3n_G + 4m_G)$. Choose the smallest $k \in \mathbb{N}_{\geq 4}$ such that

$$k \geq \alpha^{(1-\epsilon)/\epsilon}.$$

This is still polynomial in n_G and m_G while it implies that

$$k = (k^{\epsilon/(1-\epsilon)}k)^{1-\epsilon} \geq (\alpha k)^{1-\epsilon} \geq n_k^{1-\epsilon}.$$

This concludes the proof of Theorem 3.4.

B Omitted Results

B.1 Cover-Consistency

In the following two propositions, we show that neither monotonicity nor transfer-monotonicity can be dropped from the condition of Lemma 5.3. This also means that neither of the two properties implies the other.

Proposition B.1. *There exists a monotonic tournament solution S such that MoV_S does not satisfy cover-consistency.*

Proof. Let S be a tournament solution such that an alternative x is excluded if and only if it is dominated by an alternative of outdegree 1 and the tournament has size at least four.[17] Suppose that an excluded alternative x is dominated by an alternative y of outdegree 1. If x becomes dominated by an additional alternative, x remains dominated by y, whose outdegree is still 1, so it remains excluded. Hence S is monotonic.

[17] The latter condition is needed to ensure that the choice set is always nonempty.

To see that MoV_S does not satisfy cover-consistency, consider a tournament T composed of a regular tournament T' of size $2k+1 \geq 7$, along with an additional alternative x which is dominated by all alternatives in T'. Note that $S(T) = V(T)$. For any $y \in T'$, the edge (y, x) alone constitutes a DRS for y, so $\mathsf{MoV}_S(y, T) = 1$. On the other hand, since every alternative in T' has outdegree at least 3, in order for x to be dominated by an alternative of outdegree 1, at least two edges need to be reversed. This means that $\mathsf{MoV}_S(x, T) > \mathsf{MoV}_S(y, T)$ even though y covers x. □

Proposition B.2. *There exists a transfer-monotonic tournament solution S such that MoV_S does not satisfy cover-consistency.*

Proof. Let S be a tournament solution such that an alternative x is excluded if and only if it has outdegree 1 and there is another alternative of outdegree 0 (so, in particular, $D(x)$ consists only of that alternative). Assume that a tournament T' is obtained by reversing edges (y, z) and (z, x) in a tournament T. Note that in T', if x has outdegree 1, then it dominates only z which does not have outdegree 0, so $x \in S(T')$ regardless of whether $x \in S(T)$. Hence S is transfer-monotonic.

To see that MoV_S does not satisfy cover-consistency, consider any transitive tournament T. Let x and y be the alternative of outdegree 1 and 0, respectively. Then $\mathsf{MoV}_S(x, T) < 0 < \mathsf{MoV}_S(y, T)$ even though x covers y. □

B.2 Example showing that MoV_{TC} does not always follow the formula in Theorem 5.16

The tournament consists of ℓ different subtournaments $T_1, \dots, T_\ell$, each of which corresponds to a *cyclone* of size m, where m is an odd positive integer which is sufficiently larger than ℓ. A cyclone of size m is a tournament in which the m alternatives are arranged on a cycle and each alternative dominates its $(m-1)/2$ successors on the cycle. For ease of presentation, each T_i has one distinguished alternative which we call v_i. For two alternatives u and v from distinct subtournaments, say $u \in V(T_j)$ and $v \in V(T_{j'})$, it holds in general that u dominates v if and only if $j < j'$. However, there are $\ell - 1$ exceptions: all distinguished alternatives dominate v_1, i.e., $(v_2, v_1), (v_3, v_1) \dots, (v_\ell, v_1) \in E(T)$; we call these *backward edges*. See Figure 19 for an illustration.

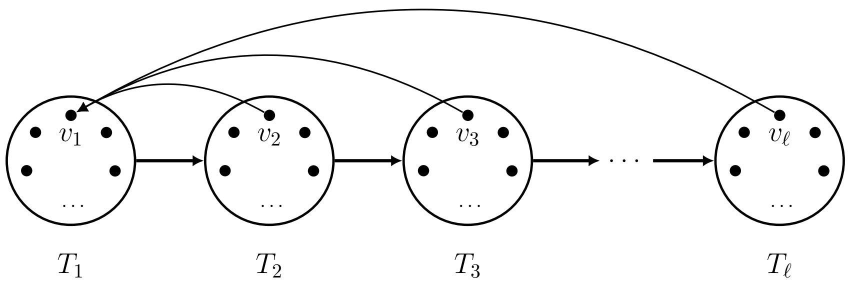

Figure 19: Illustration of the example showing that MoV_{TC} does not always follow the formula in Theorem 5.16. Each T_i is a "cyclone" of size k and has one distinguished alternative v_i.

One can check that all alternatives belong to $TC(T)$. We claim that $\mathsf{MoV}_{TC}(x, T) = \ell - (i-1)$ if $x \in V(T_i)$ for $i \geq 2$. Since reversing the backward edges $(v_i, v_1), \dots, (v_\ell, v_1)$ makes v_1 unreachable from x, we have $\mathsf{MoV}_{TC}(x, T) \leq \ell - (i-1)$. For the other direction, by Lemma 3.2, it suffices to show that even if $\ell - i$ edges are removed, x can still reach every other alternative via some directed path. Suppose that $\ell - i$ edges are removed. We first claim that in any subtournament T_j, every alternative can still reach every other alternative. Indeed, if the cyclone T_j consists of the alternatives $z_1, \dots, z_m$ in this order, then before the edges are removed, z_1 can reach $z_{(m+3)/2}$ via $(m-1)/2$ (disjoint) paths of length two; at least one of these paths remains intact after the edge removal as long as $m > 2\ell + 1$. Similarly,

$z_{(m+3)/2}$ can still reach z_2, meaning that z_1 can also reach z_2. Applying the same argument repeatedly, in T_j, every alternative can still reach every other alternative. Now, since $\ell - i$ edges are removed, one of the backward edges $(v_i, v_1), \ldots, (v_\ell, v_1)$ remains intact, say (v_s, v_1). By going to v_s via some alternative in T_s, our alternative x can then reach $v_1 \in T_1$, from where it can also reach all other alternatives in T. Hence $\mathsf{MoV}_{TC}(x, T) = \ell - (i-1)$. In particular, $\{\mathsf{MoV}_{TC}(y, T) \mid y \in V(T)\} \supseteq \{1, 2, \ldots, \ell - 1\}$. This example therefore also shows that MoV_{TC} can take on an arbitrary large number of values in a tournament.

Finally, note that the formula in Theorem 5.16 predicts a MoV value of $(m-1)/2$ for all alternatives. This can be made arbitrarily larger than $\ell - 1$ by choosing m to be as large as desired.

C Other Tournament Solutions

In this section, we present additional results for the following tournament solutions:

- The *bipartisan set* (BP) is the set of alternatives chosen with positive probability in the (unique) Nash equilibrium of the symmetric zero-sum game induced by the *tournament matrix*. In this $n \times n$ matrix, the (i, j)-entry is 1 if i dominates j, -1 if j dominates i, and 0 if $i = j$.
- The *Slater set* (SL) is the set of alternatives that appear as the Condorcet winner (i.e., maximal element) of a transitive tournament that can be obtained from the original tournament by reversing the smallest number of edges.
- The *Markov set* (MA) is the set of alternatives that win the most matches, in expectation, in a "winner-stays" tournament, where play proceeds by selecting a random alternative to replace the loser of the previous match. The Markov set corresponds to the alternatives that receive the highest probability in the (unique) eigenvector with sum of elements 1 associated with the eigenvalue 1 of the *transition matrix* of the tournament. In this $n \times n$ matrix, the (i, j)-entry is $\frac{1}{n-1}$ if i dominates j, 0 if j dominates i, and $\frac{\text{outdeg}(i)}{n-1}$ if $i = j$.
- The *minimal covering set* (MC) is the (unique) minimal UC-stable set. A set of alternatives B is said to be *UC-stable* if for every $x \in V(T) \setminus B$, the alternative x is covered in the subtournament $T|_{B \cup \{x\}}$.

Like the other tournament solutions that we have considered in this paper, all four tournament solutions are Condorcet-consistent. Moreover, all of them are contained in UC and therefore in k-kings and TC, and BP is contained in MC [Laslier, 1997, Brandt et al., 2016a]. While the bipartisan set, the Markov set, and the minimal covering set can be computed in polynomial time, deciding whether an alternative belongs to the Slater set is an NP-hard problem.

C.1 Bounds for Winners

We begin by considering MoV bounds for winners. Recall from Theorem 4.1 that for CO, TC, UC, BA, and k-kings, the MoV can be as high as $\lceil n/2 \rceil$. We demonstrate that, rather surprisingly, one edge reversal always suffices to take any alternative out of the Slater set. Indeed, this is the case even when the alternative in question is a Condorcet winner and the remaining alternatives form a regular tournament. This result also implies that computing the MoV with respect to a Slater winner can be done efficiently, and stands in contrast to the corresponding result for the Banks set (Theorem 3.5), where we used the hardness of deciding whether an alternative is a Banks winner to derive a hardness for computing the MoV of Banks winners.

Theorem C.1. *For any tournament T and any $x \in SL(T)$, we have* $\mathsf{MoV}_{SL}(x, T) = 1$.

Proof. Let $x \in SL(T)$. Consider first the case where x is a Condorcet winner. Let $y \in SL(T_{-x})$, and let r be the minimum number of edge reversals required to make T_{-x} transitive. In particular, it is possible to reverse r edges in T_{-x} and reach a transitive tournament with y as the Condorcet winner. Reverse the edge (x, y), and call the resulting tournament T'; note that $T'_{-x} = T_{-x}$. We claim that $x \notin SL(T')$. To see this, observe that in T', y can be made the Condorcet winner of a transitive tournament by reversing r edges; since x is dominated by y and dominates the remaining alternatives, it will automatically slot into the second position in the transitive order. On the other hand, in order to make x the Condorcet winner of a transitive tournament, one must make T'_{-x} transitive—this takes at least r reversals—and reverse the edge (x, y), for a total of at least $r + 1$ reversals. Hence $x \notin SL(T')$, which implies that $\mathsf{MoV}_{SL}(x, T) = 1$.

Assume now that x is not a Condorcet winner. Let T'' be the tournament obtained by making x dominate all alternatives that dominate it in T. From the previous paragraph, there exists an alternative y such that in the tournament T''_{xy} obtained by reversing the edge (x, y), we have $x \notin SL(T''_{xy})$. Recall that the Slater set is monotonic [Laslier, 1997]. If x is dominated by y in T, monotonicity implies that $x \notin SL(T)$, a contradiction. Therefore, T contains the edge (x, y). By reversing this edge, we obtain the tournament T_{xy}. Since $x \notin SL(T''_{xy})$, monotonicity implies that $x \notin SL(T_{xy})$. It follows that $\mathsf{MoV}_{SL}(x, T) = 1$, as desired. □

We now turn to the Markov set, the bipartisan set, and the minimal covering set. Since all of these tournament solutions are refinements of the uncovered set, Theorem 4.1 immediately implies the following upper bound:

Corollary C.2. *Let $S \in \{MA, BP, MC\}$. For any tournament T and any $x \in S(T)$, we have* $\mathsf{MoV}_S(x, T) \leq \lfloor n/2 \rfloor$.

We show that MoV_{MA} cannot always be as high as $\lfloor n/2 \rfloor$, at least when $n = 4$. This separates MA from CO, TC, UC, BA, and k-kings.

Proposition C.3. *Let $n = 4$. For any tournament T and any $x \in MA(T)$, we have* $\mathsf{MoV}_{MA}(x, T) = 1$.

Proof. We prove the desired statement by case analysis. Up to isomorphism, there are four tournaments with $n = 4$:

- The transitive tournament T_1 with $a_1 \succ b_1 \succ c_1 \succ d_1$.
- The tournament T_2, obtained by starting with the transitive tournament with $a_2 \succ b_2 \succ c_2 \succ d_2$ and letting d_2 dominate a_2 instead.
- The tournament T_3 with a_3 as the Condorcet winner and the remaining three alternatives forming a cycle $b_3 \succ c_3 \succ d_3 \succ b_3$.
- The tournament T_4 with d_4 as the Condorcet loser and the remaining three alternatives forming a cycle $a_4 \succ b_4 \succ c_4 \succ a_4$.

The four tournaments are depicted in Figure 20. We determine the Markov set for each of them by computing the respective probabilities that the alternatives a_i, b_i, c_i, d_i receive in the (unique) eigenvector with sum of elements 1 associated with the eigenvalue 1 in the transition matrix of T_i.

- The eigenvector for T_1 is $[1, 0, 0, 0]$, so $MA(T_1) = \{a_1\}$.
- The eigenvector for T_2 is $[4/10, 3/10, 1/10, 2/10]$, so $MA(T_2) = \{a_2\}$.
- The eigenvector for T_3 is $[1, 0, 0, 0]$, so $MA(T_3) = \{a_3\}$.
- The eigenvector for T_4 is $[1/3, 1/3, 1/3, 0]$, so $MA(T_4) = \{a_4, b_4, c_4\}$.

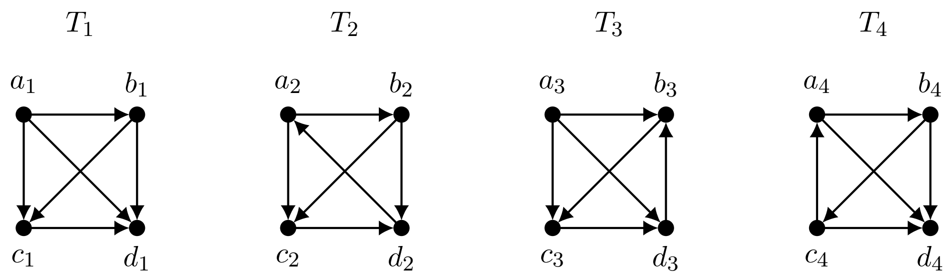

Figure 20: All four tournaments with $n = 4$ up to isomorphism.

Now, we show that for each i and each $x \in MA(T_i)$, it holds that $\mathsf{MoV}_{MA}(x, T_i) = 1$.

- For $i \in \{1, 2, 4\}$, if we reverse the edge (a_i, b_i) to obtain a tournament T_i', then b_i is a Condorcet winner in T_i', so $a_i \notin MA(T_i') = \{b_i\}$. Hence, $\mathsf{MoV}_{MA}(a_i, T_i) = 1$. Moreover, since a_4, b_4, and c_4 are symmetric in T_4, we have $\mathsf{MoV}_{MA}(b_4, T_4) = \mathsf{MoV}_{MA}(c_4, T_4) = 1$.
- For $i = 3$, if we reverse the edge (a_3, b_3), the resulting tournament T_3' is isomorphic to T_2, with the mapping $(b_3, a_3, c_3, d_3) \to (a_2, b_2, c_2, d_2)$. It follows that $a_3 \notin MA(T_3') = \{b_3\}$. Hence, $\mathsf{MoV}_{MA}(a_3, T_3) = 1$.

The proof is complete. $\square$

Our next proposition shows that for the Markov set, the MoV can be higher than 1. This means that it is not always as easy to take a winner out of the Markov set as it is for the Slater set (cf. Theorem C.1).

Proposition C.4. *For any even $n \geq 6$, there exists a tournament T and an alternative $x \in MA(T)$ such that $\mathsf{MoV}_{MA}(x, T) > 1$.*

Proof. Let $n \geq 6$ be even, and consider a tournament T in which there is a Condorcet winner x_1 and the remaining $n - 1$ alternatives $x_2, \dots, x_n$ form a "cyclone" as in Appendix B.2. We claim that $\mathsf{MoV}_{MA}(x_1, T) > 1$. If we reverse an edge (x_i, x_j) for $i, j \geq 2$, then x_1 remains the Condorcet winner and therefore a Markov winner. Hence we only need to consider the case where we reverse an edge between x_1 and one of $x_2, \dots, x_n$. Since $x_2, \dots, x_n$ are symmetric, it suffices to show that in the tournament T' obtained by reversing the edge (x_1, x_n), we have $x_1 \in MA(T')$.

Let $p_1, \dots, p_n$ denote the probabilities that $x_1, \dots, x_n$ receive in the (unique) eigenvector with sum of elements 1 associated with the eigenvalue 1 of the transition matrix of T'. The first row of this matrix implies that

$$\frac{n-2}{n-1} \cdot p_1 + \frac{1}{n-1} \cdot p_2 + \cdots + \frac{1}{n-1} \cdot p_{n-1} = p_1,$$

which is equivalent to

$$p_2 + \cdots + p_{n-1} = p_1.$$

This means that $p_1 \geq \max\{p_2, \dots, p_{n-1}\}$. If $p_1 \geq p_n$, then $x_1 \in MA(T')$ and we are done. Assume therefore that $p_n > p_1$. Since $p_1 + (p_2 + \cdots + p_{n-1}) + p_n = 1$, we have $p_n > 1/3 > p_1$. From the last row of the transition matrix of T', we have

$$\frac{1}{n-1} \cdot p_1 + \frac{1}{n-1} \cdot p_2 + \cdots + \frac{1}{n-1} \cdot p_{n/2} + \frac{n/2}{n-1} \cdot p_n = p_n,$$

which is equivalent to

$$p_1 + \cdots + p_{n/2} = \left(\frac{n}{2} - 1\right) p_n.$$

Since $p_n > 1/3$ and $p_1 + \cdots + p_{n/2} \leq 1 - p_n < 2/3$, it follows that

$$\frac{2}{3} > p_1 + \cdots + p_{n/2} = \left(\frac{n}{2} - 1\right) p_n > \frac{n-2}{6},$$

which holds only when $n < 6$. Hence, the case $p_n > p_1$ cannot occur, and the proof is complete. $\square$

For the bipartisan set, we know from Corollary C.2 that $\mathsf{MoV}_{BP}(x, T) \leq \lfloor n/2 \rfloor$ for all tournaments T and all $x \in BP(T)$. When $n = 4$, this bound is achieved by the alternative a_3 in tournament T_3 in Figure 20. While we do not know whether the bound can be attained for all n, we show that the MoV of a BP winner can be greater than 1, thereby separating the bipartisan set from the Slater set (cf. Theorem C.1). Specifically, we prove that for any tournament T with a Condorcet winner x such that no alternative has outdegree $n-2$ (such a tournament exists for any $n \geq 4$), it holds that $\mathsf{MoV}_{BP}(x, T) > 1$.

Proposition C.5. *Let T be a tournament with x as the Condorcet winner such that no alternative has outdegree $n - 2$ (in other words, the tournament T_{-x} does not have a Condorcet winner). Then* $\mathsf{MoV}_{BP}(x, T) > 1$.

To prove Proposition C.5, it is helpful to recall an alternative characterization of the bipartisan set, shown as Proposition 6.2.3 in the book by Laslier [1997].

Lemma C.6 (Laslier, 1997)**.** *For any tournament T with alternatives $x_1, \ldots, x_n$, there exists a unique probability distribution $p_1, \ldots, p_n$ over the alternatives that satisfies the following two conditions for every $i \in \{1, \ldots, n\}$:*

(i) If $p_i > 0$, then $\sum_{j: x_j \succ x_i} p_j = \sum_{j: x_i \succ x_j} p_j$;

(ii) If $p_i = 0$, then $\sum_{j: x_j \succ x_i} p_j \geq 1/2$.

The bipartisan set is then equal to the set of alternatives with positive probability, $\{x_i \mid p_i > 0\}$.

It is also known that the bipartisan set always has odd size [Laslier, 1997], a fact that we shall make use of in the following proof.

Proof of Proposition C.5. Consider a tournament T with alternatives $x_1, \ldots, x_n$ such that x_1 is a Condorcet winner and no alternative has outdegree $n - 2$. If we reverse an edge (x_i, x_j) for $i, j \geq 2$, then x_1 remains the Condorcet winner and therefore a BP winner. Hence, to establish $\mathsf{MoV}_{BP}(x_1, T) > 1$, it suffices to show that x_1 remains in the bipartisan set when we reverse an edge between it and one of the remaining alternatives.

Assume that we reverse an edge adjacent to x_1 to obtain a tournament T', and suppose for contradiction that $x_1 \notin BP(T')$. Let $p_1, \ldots, p_n$ be the probability distribution of T' satisfying the conditions in Lemma C.6. Since T does not contain any alternative with outdegree $n - 2$, there is no Condorcet winner in T'. If $BP(T')$ is a singleton, say it only contains x_i, then we must have $p_i = 1$, and condition *(ii)* implies that x_i is a Condorcet winner, a contradiction. Since $BP(T')$ has odd size, it must contain at least three alternatives, all of which receive a positive probability. If one of these alternatives receives probability at least $1/2$, then condition *(i)* cannot be satisfied with respect to any other alternative in $BP(T')$. Hence, all of the alternatives in $BP(T')$ receive probability less than $1/2$. Now, since x_1 is dominated by only one alternative in the entire tournament T', we have $\sum_{j: x_j \succ x_1} p_j < 1/2$. This contradicts condition *(ii)* with $i = 1$ and completes the proof. $\square$

Since BP is contained in MC, Proposition C.5 implies the following corollary.

Corollary C.7. *Let T be a tournament with x as the Condorcet winner such that no alternative has outdegree $n - 2$. Then* $\mathsf{MoV}_{MC}(x, T) > 1$.

C.2 Bounds for Non-Winners

We now consider MoV bounds for non-winners. Recall from Section 4.2 that the corresponding bound is constant (specifically, -1) for TC and k-kings with $k \geq 3$, logarithmic for UC and BA, and linear for CO. The following proposition places SL and MA in the same category as CO.

Proposition C.8. *Let $S \in \{SL, MA\}$. If x is a Condorcet loser in a tournament T, then $\mathsf{MoV}_S(x, T) \leq -\lfloor n/2 \rfloor$.*

Proof. It is known that every alternative in the Slater set and the Markov set has outdegree at least $\lfloor n/2 \rfloor$ [Laslier, 1997, Kim et al., 2017]. Hence, in order to bring a Condorcet loser into the Slater set or the Markov set, at least $\lfloor n/2 \rfloor$ edge reversals are necessary. □

Next, since BP and MC are contained in UC, Theorem 4.4 implies a logarithmic bound:

Proposition C.9. *For sufficiently large n, there exist a tournament T and an alternative $x \in V(T) \setminus BP(T)$ such that $\mathsf{MoV}_{BP}(x, T) \in -\Omega(\log n)$. An analogous statement holds for MC.*

If only the edges adjacent to the alternative of interest x are allowed to be reversed, then a linear number of reversals may be required to bring x into the bipartisan set.

Proposition C.10. *For any even $n \geq 4$, there exist a tournament T and an alternative $x \in V(T) \setminus BP(T)$ such that if only the edges adjacent to x can be reversed, then at least $n/2$ edges must be reversed in order to bring x into the bipartisan set of the resulting tournament.*

Proof. Let $n \geq 4$ be even, and consider a tournament T in which there is a Condorcet loser x_n and the remaining $n-1$ alternatives $x_1, \ldots, x_{n-1}$ form a regular tournament. Note that the probability distribution $(p_1, \ldots, p_n) = \left(\frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}, 0\right)$ satisfies the conditions of Lemma C.6, so $BP(T) = \{x_1, \ldots, x_{n-1}\}$. If we reverse fewer than $n/2$ edges adjacent to x_n (and no other edges), the same probability distribution still satisfies the conditions. Indeed, condition (i) remains the same, while for condition (ii) we have

$$\sum_{j: x_j \succ x_n} p_j = 1 - \sum_{j: x_n \succ x_j} p_j = 1 - |\{j \mid x_n \succ x_j\}| \cdot \frac{1}{n-1} \geq 1 - \frac{\frac{n}{2} - 1}{n-1} = \frac{n}{2n-2} > \frac{1}{2}.$$

It follows that the bipartisan set of the modified tournament remains $\{x_1, \ldots, x_{n-1}\}$. In particular, the set does not contain x_n. □

For the tournament T in the proof of Proposition C.10, it may be true that $\mathsf{MoV}_{BP}(T) = n/2$. However, in order to prove this, one would need to reason about how the bipartisan set changes as we reverse edges not adjacent to x_n; we leave it as an intriguing open question.

C.3 Structural Results

We next address structural results. We show that the margin of victory with respect to BP, MA, SL, and MC fail all of the degree-consistency notions. This leaves TC and CO as the only tournament solutions among the ones we consider to satisfy any of these notions (cf. Table 3).

Proposition C.11. *MoV_{BP}, MoV_{MA}, MoV_{SL}, and MoV_{MC} do not satisfy degree-consistency, equal-degree-consistency, and strong degree-consistency.*

Proof. It suffices to prove the claim for degree-consistency and equal-degree-consistency.

First, for each $S \in \{MA, SL\}$, there exists a tournament T such that $S(T)$ and $CO(T)$ are disjoint [Brandt et al., 2015, Table 1]. Let $x \in CO(T)$ and $y \in S(T)$. We have $\text{outdeg}(x) > \text{outdeg}(y)$ but $\mathsf{MoV}_S(x,T) < 0 < \mathsf{MoV}_S(y,T)$. It follows that MoV_{MA} and MoV_{SL} fail degree-consistency.

We establish the remaining properties with a single tournament. Consider the tournament T with five alternatives $a \succ b \succ c \succ d \succ e$ which is transitive in this order except that e dominates a.

- Since the probability distribution $(1/3, 1/3, 0, 0, 1/3)$ satisfies the conditions of Lemma C.6, $BP(T) = \{a, b, e\}$. We have $\text{outdeg}(c) > \text{outdeg}(e)$ but $\mathsf{MoV}_{BP}(c,T) < 0 < \mathsf{MoV}_{BP}(e,T)$, so MoV_{BP} fails degree-consistency. Moreover, since $\text{outdeg}(d) = \text{outdeg}(e)$ but $\mathsf{MoV}_{BP}(d,T) < 0 < \mathsf{MoV}_{BP}(e,T)$, MoV_{BP} also fails equal-degree-consistency.

- Observe that the set $\{a, b, e\}$ is UC-stable, since c is covered by b in $T|_{\{a,b,c,e\}}$ and d is covered by b in $T|_{\{a,b,d,e\}}$. Moreover, since BP is contained in MC and $BP(T) = \{a, b, e\}$, we also have $MC(T) = \{a, b, e\}$. The previous bullet point then implies that MoV_{MC} fails degree-consistency and equal-degree-consistency.

- Since the eigenvector with sum of elements 1 associated with the eigenvalue 1 in the transition matrix of T is $[3/7, 2/7, 2/21, 1/21, 1/7]$, $MA(T) = \{a\}$. We have $\text{outdeg}(a) = \text{outdeg}(b)$ but $\mathsf{MoV}_{MA}(a,T) > 0 > \mathsf{MoV}_{MA}(b,T)$, so MoV_{MA} fails equal-degree-consistency.

- In order to make a the Condorcet winner of a transitive tournament, it suffices to reverse the edge (e, a). On the other hand, for every other alternative, the analogous outcome requires reversing at least two edges. Indeed, this is clear for c, d, and e since each of them is dominated by at least two other alternatives, while for b, the tournament does not become transitive even after reversing the edge (a, b). Hence, $SL(T) = \{a\}$, and so $\mathsf{MoV}_{SL}(a,T) > 0 > \mathsf{MoV}_{SL}(b,T)$ even though $\text{outdeg}(a) = \text{outdeg}(b)$. This implies that MoV_{SL} fails equal-degree-consistency.

The proof is complete. □

In light of Proposition C.11 and Table 3, the remaining structural questions are whether MoV_{BP}, MoV_{MA}, MoV_{SL}, and MoV_{MC} satisfy cover-consistency. To answer these questions in the positive, by Lemma 5.3 and the fact that BP, MA, SL, and MC are monotonic [Laslier, 1997], it would suffice to establish that the four tournament solutions are transfer-monotonic. However, perhaps surprisingly, we show next that this is not the case for BP, SL, and MC.

Proposition C.12. *BP and MC do not satisfy transfer-monotonicity.*

Proof. Consider the tournaments T and T' as illustrated in Figure 21. We will show in the following that $x \in BP(T)$ and $x \in MC(T)$, but $x \notin BP(T')$ and $x \notin MC(T')$, where T' is derived from T by reversing the edges (z, y) and (y, x). This implies that BP and MC violate transfer-monotonicity.

We start with the argument for BP. For tournament T the unique probability distribution satisfying the two conditions in Lemma C.6 is

$$(a : 3/23,\ z : 7/23,\ b : 1/23,\ x : 1/23,\ c : 1/23,\ y : 7/23,\ d : 3/23).$$

This can be verified by checking the first condition for all alternatives:

$$\sum_{i \in D(a)} p_i = p_z + p_d = \frac{10}{23} = p_b + p_x + p_c + p_y = \sum_{i \in \overline{D}(a)} p_i;$$

$$\sum_{i \in D(z)} p_i = p_b + p_y = \frac{8}{23} = p_a + p_x + p_c + p_d = \sum_{i \in \overline{D}(z)} p_i;$$

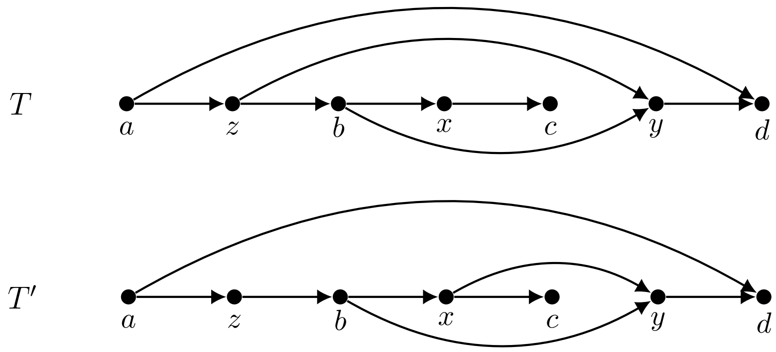

Figure 21: Illustration of the counterexample from Proposition C.12. All missing edges point from right to left.

$$\sum_{i \in D(b)} p_i = p_a + p_y + p_x = \frac{11}{23} = p_z + p_c + p_d = \sum_{i \in \overline{D}(z)} p_i;$$

$$\sum_{i \in D(x)} p_i = p_a + p_z + p_c = \frac{11}{23} = p_b + p_y + p_d = \sum_{i \in \overline{D}(x)} p_i;$$

$$\sum_{i \in D(c)} p_i = p_a + p_z + p_b = \frac{11}{23} = p_x + p_y + p_d = \sum_{i \in \overline{D}(c)} p_i;$$

$$\sum_{i \in D(y)} p_i = p_a + p_x + p_c + p_d = \frac{8}{23} = p_z + p_b = \sum_{i \in \overline{D}(y)} p_i;$$

$$\sum_{i \in D(d)} p_i = p_z + p_b + p_x + p_c = \frac{10}{23} = p_a + p_y = \sum_{i \in \overline{D}(d)} p_i.$$

Hence, $BP(T) = \{a, b, c, d, x, y, z\}$ and in particular $x \in BP(T)$. For tournament T', the probability distribution satisfying the two conditions in Lemma C.6 is

$$(a : 0,\ z : 0,\ b : 1/3,\ x : 0,\ c : 0,\ y : 1/3,\ d : 1/3).$$

To verify:

$$\sum_{i \in \overline{D}(a)} p_i = p_b + p_x + p_c + p_y = \frac{2}{3} > \frac{1}{2};$$

$$\sum_{i \in \overline{D}(z)} p_i = p_a + p_x + p_c + p_d + p_y = \frac{2}{3} > \frac{1}{2};$$

$$\sum_{i \in D(b)} p_i = p_a + p_y + p_x = \frac{1}{3} = p_z + p_c + p_d = \sum_{i \in \overline{D}(b)} p_i;$$

$$\sum_{i \in \overline{D}(x)} p_i = p_b + p_d = \frac{2}{3} > \frac{1}{2};$$

$$\sum_{i \in \overline{D}(c)} p_i = p_x + p_y + p_d = \frac{2}{3} > \frac{1}{2};$$

$$\sum_{i \in D(y)} p_i = p_a + p_z + p_c + p_d = \frac{1}{3} = p_b + p_x = \sum_{i \in \overline{D}(y)} p_i;$$

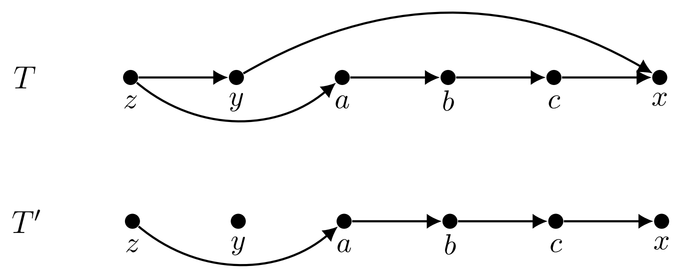

Figure 22: Illustration of the counterexample from Proposition C.13. All missing edges point from right to left.

$$\sum_{i\in D(d)} p_i = p_z + p_b + p_x + p_c = \frac{1}{3} = p_a + p_y = \sum_{i\in \overline{D}(d)} p_i.$$

Hence, $BP(T') = \{b, y, d\}$ and in particular $x \notin BP(T')$.

We turn to proving the analogous statements for MC. As BP selects a subset of MC for any tournament, we know in particular that $MC(T) = BP(T) = \{a, b, c, d, x, y, z\}$ holds. For tournament T' we will show that $MC(T') = BP(T') = \{b, y, d\}$ holds, by verifying that $\{b, y, d\}$ is UC-stable. Observe that a is covered by y in $T'|_{\{b,y,d,a\}}$, z is covered by d in $T'|_{\{b,y,d,z\}}$, x is covered by b in $T'|_{\{b,y,d,x\}}$, and c is covered by d in $T'|_{\{b,y,d,c\}}$. Hence, $MC(T') = \{b, y, d\}$ and in particular $x \notin MC(T')$.

Thus, BP and MC do not satisfy transfer-monotonicity. □

Proposition C.13. *SL does not satisfy transfer-monotonicity.*

Proof. Consider the tournaments T and T' as illustrated in Figure 22. We will show in the following that $x \in SL(T)$, but $x \notin SL(T')$, where T' is derived from T by reversing the edges (z, y) and (y, x). This implies that SL violates transfer-monotonicity.

We first show that $x \in SL(T)$. For every alternative $d \in \{a, b, c, x, y, z\}$, we write $f(d)$ for the minimum reversal distance from T to some transitive tournament having d as a Condorcet winner. To prove $x \in SL(T)$, we will show that x is a minimizer of this function. First observe that $f(z), f(y), f(a) \geq 3$ as all of these alternatives have indegree at least 3. Moreover, $f(b) \geq 3$, as alternative b has indegree 2 and, after reversing the edges (a, b) and (x, b), the tournament still contains the cycle $(y, x), (x, a), (a, y)$. Similarly, $f(c) \geq 3$, since, after reversing the only ingoing edge (b, c) of c, the tournament contains two edge-disjoint cycles, e.g., $(y, x), (x, a), (a, y)$ and $(a, b), (b, z), (z, a)$. That is, at least two additional edges have to be reversed. Lastly, we show that $f(x) \leq 3$. To see this, observe that reversing the edges $(y, x), (c, x)$, and (a, b) results in the linear order $x \succ b \succ c \succ z \succ a \succ y$. Hence, $x \in SL(T)$.

We turn to prove that $x \notin SL(T')$. For every alternative $d \in \{a, b, c, x, y, z\}$, we write $g(d)$ for the minimum reversal distance from T' to some transitive tournament having d as a Condorcet winner. We will show in the following that x is not a minimizer of this function. We first prove that $g(c) \leq 2$. To this end, observe that reversing the two edges (b, c) and (z, a) in the tournament T' leads to the linear order $c \succ x \succ a \succ b \succ y \succ z$. On the other hand, we have that $g(x) \geq 3$, as, after reversing the edge (c, x), the tournament still contains two edge-disjoint cycles, e.g., $(a, b), (b, c), (c, a)$ and $(a, y), (y, z), (z, a)$. Hence, $x \notin SL(T')$, which allows us to conclude that SL violates transfer-monotonicity. □

Despite Propositions C.12 and C.13, it could still be that MoV_{BP}, MoV_{SL}, and MoV_{MC} satisfy cover-consistency. However, we would need a new approach to prove this.

For the Markov set, our computer experiments showed that it satisfies transfer-monotonicity for all tournaments of size at most 10. Therefore, it may still be possible to establish the cover-consistency of MoV_{MA} using the same approach that we used for CO, TC, UC, k-kings, and BA.