Some problems I have proposed

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1. **Asian Pacific Mathematics Olympiad 2021, Problem 2**

   For a polynomial P and a positive integer n, define P_n as the number of positive integer pairs (a, b) such that $a < b \leq n$ and $|P(a)| - |P(b)|$ is divisible by n. Determine all polynomials P with integer coefficients such that $P_n \leq 2021$ for all positive integers n.

2. **Asian Pacific Mathematics Olympiad 2021, Problem 3**

   Let $ABCD$ be a cyclic convex quadrilateral and Γ be its circumcircle. Let E be the intersection of the diagonals AC and BD, let L be the center of the circle tangent to sides AB, BC, and CD, and let M be the midpoint of the arc BC of Γ not containing A and D. Prove that the excenter of triangle BCE opposite E lies on the line LM.

3. **International Mathematical Olympiad 2020 Shortlist, Problem C7**

   Consider any rectangular table having finitely many rows and columns, with a real number $a(r, c)$ in the cell in row r and column c. A pair (R, C), where R is a set of rows and C a set of columns, is called a *saddle pair* if the following two conditions are satisfied:

   (i) For each row r', there is $r \in R$ such that $a(r, c) \geq a(r', c)$ for all $c \in C$;

   (ii) For each column c', there is $c \in C$ such that $a(r, c) \leq a(r, c')$ for all $r \in R$.

   A saddle pair (R, C) is called a *minimal pair* if for each saddle pair (R', C') with $R' \subseteq R$ and $C' \subseteq C$, we have $R' = R$ and $C' = C$.

   Prove that any two minimal pairs contain the same number of rows.

4. **Asian Pacific Mathematics Olympiad 2020, Problem 3**

   Determine all positive integers k for which there exist a positive integer m and a set S of positive integers such that any integer $n > m$ can be written as a sum of distinct elements of S in exactly k ways.

5. **Asian Pacific Mathematics Olympiad 2020, Problem 4**

   Let $\mathbb{Z}$ denote the set of all integers. Find all polynomials $P(x)$ with integer coefficients that satisfy the following property:

For any infinite sequence $a_1, a_2, \dots$ of integers in which each integer in $\mathbb{Z}$ appears exactly once, there exist indices $i < j$ and an integer k such that $a_i + a_{i+1} + \cdots + a_j = P(k)$.

6. **International Mathematical Olympiad 2019 Shortlist, Problem C2**

   You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \le r \le 2n - 2$ you can choose a subset of the blocks whose total weight is at least r but at most $r + 2$.

7. **International Mathematical Olympiad 2019 Shortlist, Problem C8**

   Alice has a map of Wonderland, a country consisting of $n \ge 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

   Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

8. **Asian Pacific Mathematics Olympiad 2019, Problem 1**

   Let $\mathbb{Z}^+$ be the set of positive integers. Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a and b.

9. **International Mathematical Olympiad 2018 Shortlist, Problem N7** (with Pakawut Jiradilok)

   Let $n \ge 2018$ be an integer, and let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be pairwise distinct positive integers not exceeding $5n$. Suppose that the sequence

   $$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}$$

   forms an arithmetic progression. Prove that the terms of the sequence are equal.

10. **International Mathematical Olympiad 2017 Shortlist, Problem C3**

    Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

    (1) Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell.

    (2) Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell.

At the end of the game, one cell contains the number 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

11. **International Mathematical Olympiad 2017 Shortlist, Problem N3**

    Determine all integers $n \geq 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers

    $$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

    is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

12. **Asian Pacific Mathematics Olympiad 2017, Problem 1**

    We call a 5-tuple of integers *arrangeable* if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, \ldots, n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

13. **Asian Pacific Mathematics Olympiad 2017, Problem 5**
    (with Pakawut Jiradilok)

    Let n be a positive integer. A pair of n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with integer entries is called an *exquisite pair* if

    $$|a_1 b_1 + \cdots + a_n b_n| \leq 1.$$

    Determine the maximum number of distinct n-tuples with integer entries such that any two of them form an exquisite pair.

14. **International Mathematical Olympiad 2016 Shortlist, Problem N1**

    For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

    $$S(P(n)) = P(S(n)).$$

15. **Asian Pacific Mathematics Olympiad 2016, Problem 3**

    Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects the line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

16. **Asian Pacific Mathematics Olympiad 2016, Problem 4**

    The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer k such that no matter how Starways establishes its flights, the cities can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

17. **Asian Pacific Mathematics Olympiad 2015, Problem 1**

    Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point $Z \neq B$. The lines ZD and ZY intersect ω again at V and W, respectively. Prove that $AB = VW$.

18. **Asian Pacific Mathematics Olympiad 2015, Problem 4**
    (with Pakawut Jiradilok)

    Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let $\mathcal{B}$ be the set of all points on the plane that lie on at least one blue line, and $\mathcal{R}$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $\mathcal{B}$ in exactly $2n-1$ points, and also intersects $\mathcal{R}$ in exactly $2n-1$ points.

19. **Asian Pacific Mathematics Olympiad 2015, Problem 5**
    (with Pakawut Jiradilok)

    Determine all sequences $a_0, a_1, a_2, \ldots$ of positive integers with $a_0 \geq 2015$ such that for all integers $n \geq 1$:

    (i) a_{n+2} is divisible by a_n;

    (ii) $|s_{n+1} - (n+1)a_n| = 1$, where $s_{n+1} = a_{n+1} - a_n + a_{n-1} - \cdots + (-1)^{n+1}a_0$.

20. **Asian Pacific Mathematics Olympiad 2014, Problem 2**

    Let $S = \{1, 2, \ldots, 2014\}$. For each non-empty subset $T \subseteq S$, one of its members is chosen as its *representative*. Find the number of ways to assign representatives to all non-empty subsets of S so that if a subset $D \subseteq S$ is a disjoint union of non-empty subsets $A, B, C \subseteq S$, then the representative of D is also the representative of at least one of A, B, C.

21. **Asian Pacific Mathematics Olympiad 2014, Problem 3**

    Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

22. **International Mathematical Olympiad 2013, Problem 4**
    (with Potcharapol Suteparuk)

Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

23. **United States of America Mathematical Olympiad 2013, Problem 3**

    Let n be a positive integer. There are $\frac{n(n+1)}{2}$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the black side up. An *operation* is to choose a line parallel to one of the sides of the triangle, and flipping all the marks on that line. A configuration is called *admissible* if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

24. **International Mathematical Olympiad 2012 Shortlist, Problem C1**

    Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

25. **International Mathematical Olympiad 2012 Shortlist, Problem N1**

    Call admissible a set A of integers that has the following property:

    If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k.

    Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

26. **United States of America Junior Mathematical Olympiad 2012, Problem 5**

    For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \le k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

27. **International Mathematical Olympiad 2011 Shortlist, Problem A2**

    Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

    $$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$