Approximate Envy-Freeness in Graphical Cake Cutting

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Cake Cutting

- The problem of fairly allocating a divisible resource.

- Typical assumption: cake is represented by an interval $[0, 1]$.
  - Appropriate when the resource is linear.
  - Insufficient when the resource is a network.

- A more general model: Graphical Cake Cutting†

† Dividing a Graphical Cake (Bei and Suksompong, 2021)
Graphical Cake Cutting

- Resource is represented by a connected graph.
- Cake lies on the edges of the graph, can be subdivided.
- Each agent receives a connected piece of the graph.
Fairness Notions

• Common fairness notions are proportionality, maximin share, and envy-freeness.

• Graphical cake cutting:
  
  o Proportionality  (Bei and Suksompong, 2021)

  o Maximin share  (Elkind, Segal-Halevi and Suksompong, 2021)

  o Envy-freeness  (our work)

  ■ No agent would rather have another agent’s piece of cake.
Envy-Freeness

- An envy-free allocation may not exist.

- We consider approximations of envy-freeness.
  - $\alpha$-additive-EF
    - agent $i$ does not envy agent $j$ by an amount of more than $0 \leq \alpha \leq 1$.
  - $\alpha$-EF
    - agent $i$ does not envy agent $j$ by a factor of more than $\alpha \geq 1$.

- Proposition: $\alpha$-EF implies $(\alpha - 1)/(\alpha + 1)$-additive-EF.
## Our Results

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<th>Star graphs</th>
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<tbody>
<tr>
<td><strong>Non-identical valuations</strong></td>
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- Our work presents **efficient algorithms** for computing these allocations.
**General Graphs, Non-Identical Valuations**

- **Theorem**: There exists a 1/2-additive-EF allocation for a general graph.

- **Proof sketch**: Generalize ideas from interval cake cutting.†
  - Find a bundle worth **less than 1/2 to every agent and at least 1/4 to some agent**; allocate to the agent who values it at least 1/4.
    - To find this bundle, use the **DIVIDE algorithm‡** with threshold 1/4.
  - Repeat the procedure with the remaining graph and remaining agents.
  - This gives a 1/2-additive-EF allocation.

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† Contiguous Cake Cutting: Hardness Results and Approximation Algorithms (Goldberg, Hollender, and Suksompong, 2020)
‡ Dividing a Graphical Cake (Bei and Suksompong, 2021)
Star Graphs, Non-Identical Valuations

- **Theorem:** There exists a $(3 + \varepsilon)$-EF allocation for a star graph.

- **Proof sketch:** Generalize ideas from interval cake cutting.†

This gives a $(3 + \varepsilon)$-EF allocation.

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† Fair and Efficient Cake Division with Connected Pieces (Arunachaleswaran, Barman, Kumar, and Rathi, 2019)

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General Graphs, Identical Valuations

- **Theorem:** There exists a \( (2 + \varepsilon) \)-EF allocation for a general graph and agents with identical valuations.

- **Proof sketch:** Recall that we used a threshold of 1/4 previously.
  - Use an adaptive threshold of \( \frac{1}{2} \left( \frac{2i}{2n-1} - \sum_{j=1}^{i-1} \mu(A_j) \right) \) for the \( i \)th agent; this gives a 4-EF allocation.
  - Next, generalize ideas from partitioning indivisible edges of a graph.\( ^{\dagger} \)
  - Start with a 4-EF allocation and adjust adjacent bundles repeatedly; this gives a \( (2 + \varepsilon) \)-EF allocation.

\[ ^{\dagger} \text{A Tight Bound on the Min-Ratio Edge-Partitioning Problem of a Tree (Chu, Wu, Wang, and Chao, 2010)} \]

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Star Graphs, Identical Valuations

- **Theorem**: There exists a 2-EF allocation for a star graph and agents with identical valuations.

- **Proof sketch**: Use a bag-filling algorithm.

  - This gives a 2-EF allocation.

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Conclusion

● Summary

- The bounds for identical valuations are tight due to prior work.
- **Open question:** Is an $\alpha$-EF allocation guaranteed to exist for general graphs and non-identical valuations, for some constant $\alpha \geq 1$?

● Omitted results

- Settings where agents can receive a small number of connected pieces: use the notion of path similarity number.

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Thank you!

Questions?